

CLASS POLYNOMIALS FOR SOME AFFINE HECKE ALGEBRAS

ZHONGWEI YANG

ABSTRACT. Class polynomials attached to affine Hecke algebras were first introduced by X. He in [He1]. They play an important role in the study of affine Deligne-Lusztig varieties. Motivated by [He2], we compute the class polynomials attached to an affine Hecke algebra of type (twisted) A_2 . Using these class polynomials we prove a conjecture of Görtz-Haines-Kottwitz-Reuman for the general linear group, unitary group and division algebra of semisimple rank 2. Furthermore, we discuss some interesting patterns on affine Deligne-Lusztig varieties.

INTRODUCTION

In this paper, we study class polynomials of affine Hecke algebras and apply them to the study of affine Deligne-Lusztig varieties in some affine flag varieties.

Let's recall the classical Deligne-Lusztig variety first. It was introduced by Deligne and Lusztig in 1976, which was used to construct linear representations of finite groups of Lie type (see [DL]). Let \mathbb{F}_q be a finite field and \mathbf{k} be its algebraic closure. Let G be a reductive group defined over \mathbb{F}_q with a Frobenius automorphism σ and let B be a Borel subgroup defined over \mathbb{F}_q . We have the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, where W is the Weyl group. The *Deligne-Lusztig variety* associated with $w \in W$ is defined by

$$X_w = \{gB \in G/B \mid g^{-1}\sigma(g) \in BwB\}.$$

It is a locally closed subvariety of the flag variety G/B of dimension $\ell(w)$.

The notion of an affine Deligne-Lusztig variety was first introduced by Rapoport in [R], which is an analogue of Deligne and Lusztig's classical construction. For simplicity, let G be as above and let $L = \mathbf{k}((\epsilon))$ be the field of the Laurent series. Again we denote σ by the automorphism on the loop group $G(L)$. Let I be a σ -stable Iwahori subgroup of $G(L)$. We have the Iwahori-Bruhat decomposition $G(L) = \bigsqcup_{\tilde{w} \in \tilde{W}} I\tilde{w}I$ where \tilde{W} is the Iwahori-Weyl group. By definition, the *affine Deligne-Lusztig variety* $X_{\tilde{w}}(b)$ associated with $\tilde{w} \in \tilde{W}$ and $b \in G(L)$ is defined as

$$X_{\tilde{w}}(b) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in I\tilde{w}I\}.$$

The affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ plays an important role in the study of the reduction of Shimura varieties with Iwahori level structure. More precisely, it is related to the intersection of the Newton stratum and the Kottwitz-Rapoport stratum. On the special fiber of a Shimura variety there

are two important stratifications: one is the Newton stratification whose strata are indexed by certain σ -conjugacy classes $[b] \subset G(L)$; the other is the Kottwitz-Rapoport stratification whose strata are indexed by specific elements \tilde{w} of the extended affine Weyl group \tilde{W} (see [GHKR1], [H] and [R] for details).

We are particularly interested in the following questions:

- When is $X_{\tilde{w}}(b) \neq \emptyset$?
- If $X_{\tilde{w}}(b)$ is nonempty, what is $\dim X_{\tilde{w}}(b)$?

It has been studied by many other authors see [GHKR2], [GH], [GHN], [He2], [Re1], [Re2] and [V]. But an explicit answer for these questions is still unknown for general \tilde{w} and b .

In [He2], He obtained a remarkable breakthrough in the study of affine Deligne-Lusztig varieties in affine flag varieties, and He showed that the emptiness/nonemptiness pattern and dimension formula for affine Deligne-Lusztig varieties can be deduced by class polynomials of affine Hecke algebras. Thus, we can reduce questions in arithmetic geometry and number theory to questions in representation theory and Lie theory.

In this paper, we obtain the following results:

- We calculate class polynomials of the affine Hecke algebra of type (twisted) \tilde{A}_2 .
- Using class polynomials we obtain emptiness/nonemptiness pattern of $X_{\tilde{w}}(b)$ and its dimension formula for GL_3 , U_3 and \mathbb{D}_3^\times (see §3.1 for details).
- We verify a conjecture of Görtz-Haines-Kottwitz-Reuman for the general linear group, unitary group and division algebra of semisimple rank 2 (see Theorem 3.12).
- We obtain a close formula for subregular b on the number of the rational points in $X_{\tilde{w}}(b)$.
- Based on information of class polynomials and the reduction method, it is expected that the irreducible components of $X_{\tilde{w}}(b)$ of maximal dimension are controlled by the leading coefficient of the corresponding class polynomial. Then we describe the leading coefficients of corresponding class polynomials.

Here, we give a quick review of the content of this paper. In §1, we recall some definitions, e.g. Coxeter systems, (affine) Hecke algebras, loop groups, class polynomials and affine Deligne-Lusztig varieties. We also explain the algorithm of computation of class polynomials. We recall the “Dimension=Degree” theorem as well. Section §2 is the most technical part (see my PhD thesis [Y] for more detailed calculations). We first classify all the conjugacy (δ -conjugacy) classes of \tilde{W} , then we calculate the class polynomials. In §3, we apply class polynomials to the study of affine Deligne-Lusztig varieties.

1. PRELIMINARY DATA

1.1. Coxeter systems and Hecke algebras. To provide some context, let's recall Hecke algebras of Coxeter groups first. We follow [Bo], let W be a group with identity 1 and \mathbb{S} be a set of generators of W such that $\mathbb{S} = \mathbb{S}^{-1}$ and $1 \notin \mathbb{S}$. Every element of W is the product of a finite sequence of elements of \mathbb{S} . We also assume that every element of \mathbb{S} is of order 2.

Definition 1.1. (W, \mathbb{S}) is said to be a Coxeter system if it satisfies the following condition:

For $s, s' \in \mathbb{S}$, let $m_{ss'}$ be the order of ss' and let I_0 be the set of pairs (s, s') such that $m_{ss'}$ is finite. The generating set \mathbb{S} and the relations $ss'^{m_{ss'}} = 1$ for $(s, s') \in I_0$ form a presentation of the group W .

In this condition, we call W a Coxeter group. Let (W, \mathbb{S}) be a Coxeter system and $w \in W$. We recall the *length of w* (with respect to \mathbb{S}), denoted by $\ell_{\mathbb{S}}(w)$ or simply by $\ell(w)$ is the smallest integer $r \geq 0$ such that w is the product of some (or equivalently, any) sequence of r elements of \mathbb{S} .

We keep notations as in [HY] §1.1. Let H be a group of automorphisms of the group W that preserves \mathbb{S} . Set $W' = W \rtimes H$. Then an element in W' is of the form $w\delta$ for some $w \in W$ and $\delta \in H$. We have that $(w\delta)(w'\delta') = w\delta(w'\delta)\delta' \in W'$ with $\delta, \delta' \in H$. For $w \in W$ and $\delta \in H$, we set $\ell(w\delta) = \ell(w)$, where $\ell(w)$ is the length of w in the Coxeter group (W, \mathbb{S}) . Thus H consists of length 0 elements in W' .

For $J \subset \mathbb{S}$, we denote by W_J the standard parabolic subgroup of W generated by s_j for $s_j \in J$ and by W^J (resp. JW) the set of minimal coset representatives in W/W_J (resp. $W_J \backslash W$).

Let $\delta \in H$. For each δ -orbit in \mathbb{S} , we pick a single element. Let g be the product of these elements (in any order) and put $c = g\delta \in W'$. We call c a *Coxeter element* of W' (see [Spr]). Let \mathbb{O} be a δ -conjugacy class of W' , by definition \mathbb{O} is called *Coxeter* if it contains a Coxeter element of W' .

Let (W, \mathbb{S}) be a Coxeter system and R_0 be a commutative ring with 1 (in abuse of notation), and let $q \in \mathbb{C}^*$.

Definition 1.2. The Hecke algebra H (with identity T_1) associated to the Coxeter system (W, \mathbb{S}) over R_0 is the associative R_0 -algebra which is given by the following presentations:

- *Generators:* $T_s, s \in \mathbb{S}$;
- *Relations:* $T_s^2 = (q - 1)T_s + qT_1$ and $(T_s T_t)^{m_{st}} = (T_t T_s)^{m_{ts}}$ for all $s, t \in \mathbb{S}$.

1.2. Twisted loop groups. Let \mathbf{k} be the algebraic closure of a finite field \mathbb{F}_q . Let $F = \mathbb{F}_q((\epsilon))$ and $L = \mathbf{k}((\epsilon))$ be the fields of Laurent series. Let G be a connected reductive group over F and splits over a tamely ramified extension of L . Let $S \subset G$ be a maximal L -split torus defined over F , $T = Z_G(S)$ be its centralizer and N be the normalizer of T . Since \mathbf{k} is algebraically closed, G is quasi-split over L . Furthermore, T is a maximal torus.

Definition 1.3. *The algebraic loop group LG associated with G is the ind-group scheme over \mathbf{k} representing the functor*

$$R \longmapsto LG(R) = G(R((\epsilon)))$$

on the category of \mathbf{k} -algebras.

Let $\sigma \in \text{Gal}(L/F)$ be the Frobenius automorphism. It induces an automorphism on $G(L)$, and we denote the induced automorphism by the same symbol.

Let \mathbb{A} be the *apartment* of $G(L)$ corresponding to S and \mathfrak{a}_C be a σ -invariant alcove in \mathbb{A} . Let $I \subset G(L)$ be the Iwahori subgroup corresponding to \mathfrak{a}_C over L and $\widetilde{\mathbb{S}}$ be the set of simple reflections at the walls of \mathfrak{a}_C .

By definition, the *finite Weyl group* W associated with S is

$$W = N(L)/T(L)$$

and the *Iwahori-Weyl group* \widetilde{W} associated with S is

$$\widetilde{W} = N(L)/T(L)_1$$

where $T(L)_1$ is the unique parahoric subgroup of $T(L)$. Since $\widetilde{W} \cong I \backslash G(L) / I$, we embed \widetilde{W} set-theoretically into $G(L)$. Thus we identify representatives of \widetilde{W} in $G(L)$ with elements in \widetilde{W} . We have the *Iwahori-Bruhat decomposition*

$$G(L) = \bigsqcup_{\tilde{w} \in \widetilde{W}} I \tilde{w} I.$$

Let Γ be the absolute Galois group $\text{Gal}(\bar{L}/L)$ and P be the Γ -coinvariants of $X_*(T)$. By [HR], and by choosing a special vertex in \mathbb{A} we identify $T(L)/T(L)_1$ with P . We also obtain a split short exact sequence $1 \longrightarrow P \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 1$ and a semi-direct product $\widetilde{W} = P \rtimes W$. The automorphism σ on $G(L)$ induces an automorphism on \widetilde{W} and we denote it by δ . The map gives a bijection on $\widetilde{\mathbb{S}}$. We choose a special vertex in \mathbb{A} such that the previous split short exact sequence is preserved by δ . Thus δ induces an automorphism on W and we denote it by the same symbol.

Let Φ be the set of roots of (G, S) over L and Φ_a be the set of affine roots. Let \mathbb{S} be the set of simple roots in Φ . We identify \mathbb{S} with the set of simple reflections in W , then \mathbb{S} is a δ -stable proper subset of $\widetilde{\mathbb{S}}$. We denote by Φ^+ the set of positive roots of Φ and ρ the half of the sum of all positive roots in Φ .

Let G_1 be the subgroup of $G(L)$ generated by all parahoric subgroups. We take $N_1 = N(L) \cap G_1$, then by [BT], the quadruple $(G_1, I, N_1, \widetilde{\mathbb{S}})$ is a double Tits system with affine Weyl group

$$W_a = N_1 / (N(L) \cap I).$$

We identify W_a to the Iwahori-Weyl group of the simply connected cover G_{sc} of the the derived group G_{der} of G . Let T_{sc} be the maximal torus of G_{sc} giving by T . Thus we have $W_a = X_*(T_{sc})_\Gamma \rtimes W$. It showed that there exists a reduced root system Δ such that $W_a = Q^\vee(\Delta) \rtimes W(\Delta)$, where $Q^\vee(\Delta)$ is the

coroot lattice of Δ . We write Q for $Q^\vee(\Delta)$ and identify Q with $X_*(T_{sc})_\Gamma$ and $W(\Delta)$ with W .

Remark 1.4. *The pairs (W, \mathbb{S}) and $(W_a, \widetilde{\mathbb{S}})$ are Coxeter systems. Thus we already have length functions on W and W_a (see §1.1). But the Iwahori-Weyl group \widetilde{W} is not a Coxeter group. For any element $\tilde{w} \in \widetilde{W}$, the length of \tilde{w} (denoted as $\ell(\tilde{w})$) is the number of “affine root hyperplanes” between $\tilde{w}(\alpha_C)$ and α_C in \mathbb{A} . Let Ω be the subgroup of \widetilde{W} consisting length 0 elements. The Iwahori-Weyl group \widetilde{W} is a quasi-Coxeter group in the sense that $\widetilde{W} = W_a \rtimes \Omega$.*

1.3. Class polynomials. Analogizing the definition of a Hecke algebra associated to a Coxeter system, we recall a Hecke algebra associated with a quasi-Coxeter system.

Definition 1.5. *Let \widetilde{H} be the Hecke algebra associated with \widetilde{W} , i.e., \widetilde{H} is the associative $A = \mathbb{Z}[v, v^{-1}]$ -algebra with basis $T_{\tilde{w}}$ for $\tilde{w} \in \widetilde{W}$ and multiplication is given by*

$$\begin{aligned} T_{\tilde{x}}T_{\tilde{y}} &= T_{\tilde{x}\tilde{y}}, & \text{if } \ell(\tilde{x}) + \ell(\tilde{y}) &= \ell(\tilde{x}\tilde{y}); \\ (T_s - v)(T_s + v^{-1}) &= 0, & \text{for } s &\in \widetilde{\mathbb{S}}. \end{aligned}$$

Note that the map $T_{\tilde{w}} \mapsto T_{\delta(\tilde{w})}$ defines an A -algebra automorphism of \widetilde{H} , and we still denote it as δ .

For any $\tilde{w}, \tilde{w}' \in \widetilde{W}$ are said to be δ -conjugate if $\tilde{w}' = \tilde{x}\tilde{w}\delta(\tilde{x})^{-1}$ for some $\tilde{x} \in \widetilde{W}$. For $\tilde{w}, \tilde{w}' \in \widetilde{W}$ and $s_i \in \widetilde{\mathbb{S}}$, we write $\tilde{w} \xrightarrow{s_i}_\delta \tilde{w}'$ if $\tilde{w}' = s_i\tilde{w}s_{\delta(i)}$ and $\ell(\tilde{w}') \leq \ell(\tilde{w})$. We write $\tilde{w} \rightarrow_\delta \tilde{w}'$ if there is a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n = \tilde{w}'$ of elements in \widetilde{W} such that for any k , $\tilde{w}_{k-1} \xrightarrow{s_i}_\delta \tilde{w}_k$ for some $s_i \in \widetilde{\mathbb{S}}$. We write $\tilde{w} \approx_\delta \tilde{w}'$ if $\tilde{w} \rightarrow_\delta \tilde{w}'$ and $\tilde{w}' \rightarrow_\delta \tilde{w}$. Write $\tilde{w} \approx_\delta \tilde{w}'$ if $\tilde{w} \approx_\delta \tau\tilde{w}'\delta(\tau)^{-1}$ for some $\tau \in \Omega$.

We say that $\tilde{w}, \tilde{w}' \in \widetilde{W}$ are elementarily strongly δ -conjugate if $\ell(\tilde{w}) = \ell(\tilde{w}')$ and there exists $\tilde{x} \in \widetilde{W}$ such that $\tilde{w}' = \tilde{x}\tilde{w}\delta(\tilde{x})^{-1}$ and $\ell(\tilde{x}\tilde{w}) = \ell(\tilde{x}) + \ell(\tilde{w})$ or $\ell(\tilde{w}\delta(\tilde{x})^{-1}) = \ell(\tilde{x}) + \ell(\tilde{w})$. And we call that \tilde{w}, \tilde{w}' are strongly δ -conjugate if there is a sequence $\tilde{w} = \tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n = \tilde{w}'$ of elements in \widetilde{W} such that for any i , \tilde{w}_{i-1} is elementarily strongly δ -conjugate to \tilde{w}_i . We write $\tilde{w} \sim_\delta \tilde{w}'$ if \tilde{w} and \tilde{w}' are strongly δ -conjugate.

In [HN], He and Nie proved that minimal length elements w_\circ of any δ -conjugacy class \circ of \widetilde{W} satisfy some special properties, generalizing the results of Geck and Pfeiffer [GP] on finite Weyl groups. These properties play a key role in the study of affine Deligne-Lusztig varieties and affine Hecke algebras. In [He2] and [HN], it is showed that for any δ -conjugacy class \circ , we can fix a minimal length representative w_\circ and the image of T_{w_\circ} in $\widetilde{H}/[\widetilde{H}, \widetilde{H}]_\delta$ where $[\widetilde{H}, \widetilde{H}]_\delta$ is the A -submodule (we regard \widetilde{H} as a left A -module) of \widetilde{H} generated by all δ -commutators (i.e. for any $h, h' \in \widetilde{H}$, $[h, h']_\delta = hh' - h'\delta(h)$). Moreover, T_{w_\circ} is independent of the choice of w_\circ and forms a basis of $\widetilde{H}/[\widetilde{H}, \widetilde{H}]_\delta$. It is proved in [HN, Theorem 6.7] that

Theorem 1.6 (He-Nie). *The elements $T_{\tilde{w}_\circ}$ form an A -basis of $\tilde{H}/[\tilde{H}, \tilde{H}]_\delta$, here \circ runs over all the δ -conjugacy classes of \tilde{W} .*

From now on we denote $T_{\tilde{w}_\circ}$ as T_\circ for simplicity. For any $\tilde{w} \in \tilde{W}$ and a δ -conjugacy class \circ , there exists a unique $f_{\tilde{w}, \circ} \in A$ such that

$$T_{\tilde{w}} \equiv \sum_{\circ} f_{\tilde{w}, \circ} T_\circ \pmod{[\tilde{H}, \tilde{H}]_\delta}.$$

$f_{\tilde{w}, \circ}$ is a polynomial in $\mathbb{Z}[v - v^{-1}]$ with nonnegative coefficient. This is called the **class polynomial** attached to \tilde{w} and \circ , and it can be constructed inductively as follows:

If \tilde{w} is a minimal length element in a δ -conjugacy class of \tilde{W} , then we set

$$f_{\tilde{w}, \circ} = \begin{cases} 1, & \text{if } \tilde{w} \in \circ, \\ 0, & \text{if } \tilde{w} \notin \circ. \end{cases}$$

If \tilde{w} is not a minimal length element in its δ -conjugacy class and for any $\tilde{w}' \in \tilde{W}$ with $\ell(\tilde{w}') < \ell(\tilde{w})$, $f_{\tilde{w}', \circ}$ is constructed. By [HN], there exists $\tilde{w}_1 \approx_\delta \tilde{w}$ and $s_i \in \tilde{\mathbb{S}}$ such that $\ell(s_i \tilde{w}_1 s_{\delta(i)}) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. In this case, $\ell(s_i \tilde{w}) < \ell(\tilde{w})$ and we define $f_{\tilde{w}, \circ}$ as

$$f_{\tilde{w}, \circ} = (v - v^{-1})f_{s_i \tilde{w}_1, \circ} + f_{s_i \tilde{w}_1 s_{\delta(i)}, \circ}.$$

1.4. The “Dimension=Degree” theorem. We keep the notations as before. Let $\mathbf{Fl} = G(L)/I$ be the *fppf* quotient, then \mathbf{Fl} is represented by an ind-scheme, ind-projective over \mathbf{k} . We also have the Iwahori-Bruhat decomposition:

$$\mathbf{Fl} = \bigsqcup_{\tilde{w} \in \tilde{W}} I\tilde{w}I/I.$$

Definition 1.7. For $\tilde{w} \in \tilde{W}$ and $b \in G(L)$, the affine Deligne-Lusztig variety associated with \tilde{w} and b is the locally closed sub-ind scheme $X_{\tilde{w}}(b)(\mathbf{k})$ in the affine flag variety \mathbf{Fl} defined as

$$X_{\tilde{w}}(b)(\mathbf{k}) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in I\tilde{w}I\}.$$

Remark 1.8. The affine Deligne-Lusztig variety $X_{\tilde{w}}(b)(\mathbf{k})$ is a finite-dimensional \mathbf{k} -scheme and locally of finite type over \mathbf{k} .

We call an element $\tilde{w} \in \tilde{W}$ a δ -straight element if and only if for any $n \in \mathbb{N}$ we have $\ell(\tilde{w}\delta(\tilde{w}) \cdots \delta^{n-1}(\tilde{w})) = n\ell(\tilde{w})$. We call a δ -conjugacy class in \tilde{W} straight if it contains some straight element. We denote by \cdot_σ the σ -conjugation action on $G(L)$ and it is defined by that for any $g, g' \in G(L)$, $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$. We have the following Kottwitz’s classification of σ -conjugacy classes $B(G)$ on $G(L)$.

Theorem 1.9 (Kottwitz, He). *For any straight δ -conjugacy class \circ of \tilde{W} , we fix a minimal length representative \tilde{w}_\circ . Then*

$$G(L) = \bigsqcup_{\circ} G(L) \cdot_\sigma \tilde{w}_\circ,$$

here \mathbb{O} runs over all the straight δ -conjugacy classes of \widetilde{W} .

Let $(P/Q)_\delta$ be the δ -coinvariants on P/Q , let

$$\kappa : \widetilde{W} \longrightarrow \widetilde{W}/W_a \cong P/Q \longrightarrow (P/Q)_\delta$$

be the natural projection. We call κ the *Kottwitz map*.

Let $P_{\mathbb{Q}} = P \otimes_{\mathbb{Z}} \mathbb{Q}$ and $P_{\mathbb{Q}}/W$ be the quotient of $P_{\mathbb{Q}}$ by the natural action of W . We can identify $P_{\mathbb{Q}}/W$ with $P_{\mathbb{Q},+} = \{\chi \in P_{\mathbb{Q}} \mid \alpha(\chi) \geq 0, \text{ for all } \alpha \in \Phi^+\}$. Let $P_{\mathbb{Q},+}^\delta$ be the set of δ -invariant points in $P_{\mathbb{Q},+}$.

Since the image of $W \rtimes \langle \delta \rangle$ in $\text{Aut}(W)$ is a finite group, for each $\tilde{w} = t^\chi w \in \widetilde{W}$, there exists $n \in \mathbb{N}$ such that $\delta^n = 1$ and $w\delta(w)\delta^2(w) \cdots \delta^{n-1}(w) = 1$. Then $\tilde{w}\delta(\tilde{w})\delta^2(\tilde{w}) \cdots \delta^{n-1}(\tilde{w}) = t^\lambda$ for some $\lambda \in P$. Let $v_{\tilde{w}} = \lambda/n \in P_{\mathbb{Q}}$ and $\bar{v}_{\tilde{w}}$ be the corresponding element in $P_{\mathbb{Q},+}$. Note that $v_{\tilde{w}}$ is independent of the choice of n . Let $\bar{v}_{\tilde{w}}$ be the unique element in $P_{\mathbb{Q},+}$ that lies in the W -orbit of $v_{\tilde{w}}$. Since $t^\lambda = \tilde{w}t^{\delta(\lambda)}\tilde{w}^{-1} = t^{w\delta(\lambda)}$, $\bar{v}_{\tilde{w}} \in P_{\mathbb{Q},+}^\delta$. We call the map $\widetilde{W} \longrightarrow P_{\mathbb{Q},+}^\delta$ with $\tilde{w} \mapsto \bar{v}_{\tilde{w}}$ the *Newton map*. We define

$$f : \widetilde{W} \longrightarrow P_{\mathbb{Q},+}^\delta \times (P/Q)_\delta \quad \text{by} \quad \tilde{w} \longmapsto (\bar{v}_{\tilde{w}}, \kappa(\tilde{w})).$$

Follows [K], f is constant on each δ -conjugacy class of \widetilde{W} . We denote the image of the map by $B(\widetilde{W}, \delta)$.

Remark 1.10. (1) There is a bijection between the set of σ -conjugacy classes $B(G)$ of $G(L)$ and the set of straight conjugacy classes of \widetilde{W} .

(2) Any σ -conjugacy class of $G(L)$ contains a representative in \widetilde{W} . Moreover, the map $f : \widetilde{W} \longrightarrow P_{\mathbb{Q},+}^\delta \times (P/Q)_\delta$ is in fact the restriction of a map defined on $G(L)$ as $b \mapsto (\bar{v}_b, \kappa(b))$ and we call \bar{v}_b the *Newton vector* of b .

The “Dimension=Degree” theorem is a main result in [He2], we quote it as follows:

Theorem 1.11 (He). *Let $b \in G(L)$ and $\tilde{w} \in \widetilde{W}$. Then*

$$\dim(X_{\tilde{w}}(b)) = \max_{\mathbb{O}} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}) + \deg(f_{\tilde{w}, \mathbb{O}})) - \langle \bar{v}_b, 2\rho \rangle,$$

here \mathbb{O} runs over δ -conjugacy classes of \widetilde{W} with $f(\mathbb{O}) = f(b)$ and $\ell(\mathbb{O})$ is the length of any minimal length element in \mathbb{O} .

Remark 1.12. We use the convention that the dimension of an empty variety and the degree of a zero polynomial are both $-\infty$.

Theorem 1.11 relates the dimension of affine Deligne-Lusztig varieties to the degree of the class polynomials which provides both the theoretic and practical way to determine the dimension of affine Deligne-Lusztig varieties. It shows that the dimension and emptiness/nonemptiness pattern of affine Deligne-Lusztig varieties $X_{\tilde{w}}(b)$ only depend on the data $(\widetilde{W}, \delta, \tilde{w}, f(b))$ and thus independent of the choice of G . And it implies that the emptiness/nonemptiness and dimension formula of affine Deligne-Lusztig varieties only rely on the reduction method. This theorem inspired my study of the class polynomials of affine Hecke algebras.

2. CLASS POLYNOMIALS FOR AFFINE HECKE ALGEBRAS OF TYPE (TWISTED) \widetilde{A}_2

We calculate class polynomials for affine Hecke algebras of type \widetilde{A}_2 for split groups of adjoint type (PGL_3). We let \widetilde{W} be the Iwahori-Weyl group, then $\widetilde{W} = P \rtimes W = W_a \rtimes \Omega$ where P is the coweight lattice and W_a is the affine Weyl group generated by the simple reflections $\widetilde{S} = \{s_0, s_1, s_2\}$. Here $\Omega = \langle \tau \rangle$ with $\tau^3 = 1$ and $\tau s_0 \tau^{-1} = s_1$, $\tau s_1 \tau^{-1} = s_2$, $\tau s_2 \tau^{-1} = s_0$. Let α_1 and α_2 be the corresponding simple roots.

Before we calculate class polynomials, let me quote the following lemma [HN, Lemma 5.1] which will be heavily used in the following computations.

Lemma 2.1 (He-Nie). *Let $\tilde{w}, \tilde{w}' \in \widetilde{W}$ with $\tilde{w} \sim \tilde{w}'$. Then*

$$T_{\tilde{w}} \equiv T_{\tilde{w}'} \pmod{[\widetilde{H}, \widetilde{H}]}.$$

As you will see that the whole section is the most technique part in this paper, we'd like to make some comments here. It is quite hard to calculate class polynomials and we know little about them in general. And till now, we have found neither an efficient nor a systematic way to calculate them in general situations. Even in the following low rank cases (of type (twisted) \widetilde{A}_2), the calculations are quite difficult and highly nontrivial. From the definition of a class polynomial $f_{\tilde{w}, \mathbb{O}}$, two parameters are involved: an extended affine Weyl group element \tilde{w} and a $(\delta-)$ conjugacy class \mathbb{O} in \widetilde{W} . As we know that for any \tilde{w} we can be weakly reduced to \tilde{w}' i.e. $(\tilde{w} \rightarrow_{\delta} \tilde{w}') \tilde{w} \rightarrow \tilde{w}'$, where \tilde{w}' is a minimal length element in the $(\delta-)$ conjugacy class \mathbb{O} of \tilde{w} [HN]. To calculate class polynomials, we will by way of doing the reduction procedure, and thus there are quite lot paths involved which cause enormous complexities and extreme difficulties. There is an illusion that the calculations in the coming subsections looks routine. Since for different \tilde{w} we have to work hard on choosing correct path to make the calculations work, actually, those calculations are very subtle and difficult and you will see how in the following subsections.

2.1. Class polynomials for elements in W_a . We first classify all \widetilde{W} -conjugacy classes in W_a .

Lemma 2.2. *Note that all elements in W_a with length 1 are \widetilde{W} -conjugate. Let \mathbb{O}_1 be the \widetilde{W} -conjugacy class in W_a with minimal length 1. Then*

$$\mathbb{O}_1 = \{t^{k\alpha_1} s_1, t^{k\alpha_2} s_2, t^{k(\alpha_1+\alpha_2)} s_1 s_2 s_1 \mid k \in \mathbb{Z}\}.$$

Lemma 2.3. *All elements in W_a with minimal length 2 are \widetilde{W} -conjugate. Let \mathbb{O}_2 be the \widetilde{W} -conjugacy class in W_a with minimal length 2. Then*

$$\mathbb{O}_2 = \{t^{\lambda} s_1 s_2, t^{\lambda} s_2 s_1 \mid \lambda \in Q\}.$$

Let $Q_{sh} = Q - \{k(\alpha_1 + 2\alpha_2), k(2\alpha_1 + \alpha_2), k(\alpha_1 - \alpha_2) \mid k \in \mathbb{Z}\}$.

Lemma 2.4. *For any $\lambda \in P_+ \cap Q_{sh}$, i.e. $\lambda = m\alpha_1 + n\alpha_2$ where $m, n \in \mathbb{Z}$ and $1 \leq m \leq n \leq 2m-1$ or $1 \leq n \leq m \leq 2n-1$. We set $\mathbb{O}_{\lambda} = \widetilde{W} \cdot t^{\lambda}$, then*

$$\mathbb{O}_{\lambda} = \{t^{m\alpha_1+n\alpha_2}, t^{(n-m)\alpha_1+n\alpha_2}, t^{m\alpha_1+(m-n)\alpha_2}, t^{-n\alpha_1-m\alpha_2}, t^{-n\alpha_1+(m-n)\alpha_2}, t^{(n-m)\alpha_1-m\alpha_2}\},$$

with $\ell(\mathbb{O}_\lambda) = \ell(t^\lambda)$. If $\lambda = m(\alpha_1 + 2\alpha_2)$, then

$$\mathbb{O}_\lambda = \{t^{m(\alpha_1+2\alpha_2)}, t^{-m(2\alpha_1+\alpha_2)}, t^{m(\alpha_1-\alpha_2)}\},$$

or if $\lambda = m(2\alpha_1 + \alpha_2)$, then

$$\mathbb{O}_\lambda = \{t^{m(2\alpha_1+\alpha_2)}, t^{-m(\alpha_1+2\alpha_2)}, t^{-m(\alpha_1-\alpha_2)}\},$$

with $\ell(\mathbb{O}_\lambda) = \ell(t^\lambda)$.

Lemma 2.5. For $i \in \mathbb{N}_+$, set $\mathbb{C}_i = \{t^{k\alpha_1+i\alpha_2}s_1, t^{(-i)\alpha_1+k\alpha_2}s_2, t^{k\alpha_1+(k-i)\alpha_2}s_1s_2s_1 \mid k \in \mathbb{Z}\}$ and $\mathbb{C}'_i = \{t^{k\alpha_1-i\alpha_2}s_1, t^{i\alpha_1+k\alpha_2}s_2, t^{(k-i)\alpha_1+k\alpha_2}s_1s_2s_1 \mid k \in \mathbb{Z}\}$. Then \mathbb{C}_i and \mathbb{C}'_i are \widetilde{W} -conjugacy classes of W_a with minimal length $3i$ if i is odd or with minimal length $3i + 1$ if i is even.

The proofs of the above four lemmas 2.2, 2.3, 2.4 and 2.5 are direct and methods are quite similar to each other. It is enough for us to give a proof of 2.3 as an example, and we omit the others.

Proof of Lemma 2.3. Let $\mathbb{O}'_2 = \{t^\lambda s_1s_2, t^\lambda s_2s_1 \mid \lambda \in Q\}$ it is quite easy to show that $\widetilde{W} \cdot \mathbb{O}'_2 \subset \mathbb{O}'_2$. Now we prove that for any element $\tilde{w} \in \mathbb{O}'_2$, then \tilde{w} is \widetilde{W} -conjugate to s_0s_1 . We use induction on length $\ell(\tilde{w})$. Assume for any $\tilde{w}' \in \mathbb{O}'_2$ and $\ell(\tilde{w}') < \ell(\tilde{w})$, then \tilde{w}' is \widetilde{W} -conjugate to s_0s_1 . We know \tilde{w} can be written uniquely as $xt^\mu y$ where $x \in W$, $\mu \in Q \cap P_+$, $y \in {}^{I(\mu)}W$, here $I(\mu) = \{s_i \in \mathbb{S} \mid \langle \mu, \alpha_i \rangle = 0\}$. If $\tilde{w} = xt^\mu y$, $x = s_{i_1} \cdots s_{i_r} \neq 1$ (reduced expression) with $s_{i_j} \in \mathbb{S}$ for $1 \leq j \leq r$. Then set $\tilde{w}_1 = s_{i_1}\tilde{w}s_{i_1} = s_{i_2} \cdots s_{i_r} t^\mu y s_{i_1}$, then

$$\ell(\tilde{w}_1) = \ell(\mu) + \ell(x) - 1 - \ell(y s_{i_1}) \leq \ell(\mu) + \ell(x) - \ell(y) = \ell(\tilde{w}).$$

If $\ell(\tilde{w}_1) < \ell(\tilde{w})$, by induction we are done. If $\ell(\tilde{w}_1) = \ell(\tilde{w})$, we set $\tilde{w}_2 = s_{i_2}\tilde{w}_1s_{i_2} = s_{i_3} \cdots s_{i_r} t^\mu y s_{i_1} s_{i_2}$ and

$$\ell(\tilde{w}_2) = \ell(\mu) + \ell(x) - 2 - \ell(y s_{i_1} s_{i_2}) \leq \ell(\mu) + \ell(x) - 1 - \ell(y s_{i_1}) = \ell(\tilde{w}_1).$$

By the same argument, we can reduce to the case $\tilde{w} = t^\mu s_1s_2$ or $t^\mu s_2s_1$ with $\mu \in Q \cap P_+$. Now if $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_1s_2$ with $1 \leq m \leq n < 2m - 1$ or $1 \leq n < m \leq 2n$. If $\ell(\tilde{w}) = 2$, obvious. If $\ell(\tilde{w}) > 2$, set $\tilde{w}_1 = s_0\tilde{w}s_0 = t^{(1-n)\alpha_1+(2-m)\alpha_2}s_2s_1$, we have $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$. By induction \tilde{w}_1 is \widetilde{W} -conjugate to s_0s_1 , so is \tilde{w} . For $\tilde{w} = t^{k\alpha_1+2k\alpha_2}s_1s_2$ ($k \geq 1$), check directly $\tilde{w}_1 = s_1\tilde{w}s_1$, then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, which deduce that \tilde{w} is \widetilde{W} -conjugate to s_0s_1 . For $\tilde{w} = t^{k\alpha_1+(2k-1)\alpha_2}s_1s_2$ then $\tilde{w}_1 = s_1s_0\tilde{w}s_0s_1$ and $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$. By induction, \tilde{w}_1 is \widetilde{W} -conjugate to s_0s_1 , so is \tilde{w} . If $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_2s_1$ with $1 \leq m \leq n \leq 2m$ or $1 \leq n < m < 2n - 1$. If $\ell(\tilde{w}) = 2$, obvious. If $\ell(\tilde{w}) > 2$, set $\tilde{w}_1 = s_0\tilde{w}s_0 = t^{(2-n)\alpha_1+(1-m)\alpha_2}s_1s_2$. Also $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$. By induction, \tilde{w}_1 is \widetilde{W} -conjugate to s_0s_1 , so is \tilde{w} . For $\tilde{w} = t^{2k\alpha_1+k\alpha_1}s_2s_1$ ($k \geq 1$), check directly $\tilde{w}_1 = s_2\tilde{w}s_2$, then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, which deduce \tilde{w} is \widetilde{W} -conjugate to s_0s_1 . For $\tilde{w} = t^{(2k-1)\alpha_1+k\alpha_2}s_2s_1$ ($k \geq 1$), then $\tilde{w}_1 = s_2s_0\tilde{w}s_0s_2$ and $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$. By induction, \tilde{w}_1 is \widetilde{W} -conjugate to s_0s_1 , so is \tilde{w} . \square

Theorem 2.6. The \widetilde{W} -conjugacy classes in W_a are as following:

$$\{Id\}, \mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_\lambda, \mathbb{C}_i, \text{ and } \mathbb{C}'_i$$

where $\lambda \in P_+ \cap Q$, and $i \in \mathbb{N}_+$.

Proof. Following Lemmas 2.2, 2.3, 2.4, 2.5 and

$$W_a = \{Id\} \sqcup \mathbb{O}_1 \sqcup \mathbb{O}_2 \sqcup (\sqcup_{\lambda \in P_+ \cap Q} \mathbb{O}_\lambda) \sqcup (\sqcup_{i \in \mathbb{N}_+} (\mathbb{C}_i \sqcup \mathbb{C}'_i)),$$

we obtain the theorem directly. \square

Before the calculating of class polynomials, for convenience, we fix some notations. We set $\mathbb{I} = \{t^{m\alpha_1+n\alpha_2} w \mid m \in \mathbb{N}_+, n \in \mathbb{Z}, 1-m \leq n \leq 2m-1, w \in W\} \sqcup \{t^{k\alpha_1+2k\alpha_2}, t^{k\alpha_1+2k\alpha_2}s_2, t^{k\alpha_1+2k\alpha_2}s_2s_1, t^{k(\alpha_1-\alpha_2)}, t^{k(\alpha_1-\alpha_2)}s_1, t^{k(\alpha_1-\alpha_2)}s_2 \mid k \in \mathbb{N}_+\} \sqcup \{Id, s_2\}$. Since $W_a = \mathbb{I} \cup (\tau \cdot \mathbb{I}) \cup (\tau^2 \cdot \mathbb{I})$ and 2.1, it is sufficient to consider elements in \mathbb{I} . For $\alpha \in Q$ we set $\underline{Q}_\alpha = \{\lambda \in Q \cap P_+ \mid \lambda < \alpha\} \cap Q_{sh}$. For $\lambda \in Q \cap P_+$, let $\mathbb{C}(t^\lambda s_1)$ to be the \widetilde{W} -conjugacy class contains $t^\lambda s_1$, and let $\mathbb{O}_{t^\lambda s_1}^\leq$ to be the set of all \mathbb{C}_i with $\ell(\mathbb{C}_i) \leq \ell(\mathbb{C}(t^\lambda s_1))$. Similarly, let $\mathbb{C}'(t^\lambda s_2)$ to be the \widetilde{W} -conjugacy class contains $t^\lambda s_2$, and let $\mathbb{O}_{t^\lambda s_2}^{\prime \leq}$ to be the set of all \mathbb{C}'_i with $\ell(\mathbb{C}'_i) \leq \ell(\mathbb{C}'(t^\lambda s_2))$. We calculate directly and obtain the following class polynomials. In the following, since the computations are quite similar, we list some typical calculations and others are easily obtained by similar methods. For other detailed calculations see [Y, Chapter 3].

Proposition 2.7. *If $\tilde{w} \in \mathbb{O}_\lambda$ where $\lambda \in P_+ \cap Q$. Then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} 1, & \mathbb{O} = \mathbb{O}_\lambda \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since \mathbb{O}_λ only has finitely many elements for each λ and any two elements in \mathbb{O}_λ are \sim , the proposition follows directly from the definition of class polynomials. \square

In the following sections, for any $\tilde{w}, \tilde{w}' \in \widetilde{W}$, we will always write

$$T_{\tilde{w}} \equiv T_{\tilde{w}'} \pmod{[\widetilde{H}, \widetilde{H}]} \text{ (or } [\widetilde{H}, \widetilde{H}]_\delta \text{)}$$

as

$$T_{\tilde{w}} \equiv T_{\tilde{w}'}$$

for short.

Proposition 2.8. (1) *If $\tilde{w} = t^{k(\alpha_1+2\alpha_2)}s_2s_1$, with $k \in \mathbb{N}_+$, then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{(k-1)\alpha_1+(2k-2)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{k\alpha_1+(2k-1)\alpha_2}s_1}^\leq \cup \mathbb{O}_{t^{(k-1)\alpha_1+(2k-2)\alpha_2}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If $\tilde{w} = t^{k(\alpha_1+2\alpha_2)}s_1s_2$, with $k \in \mathbb{N}_+$, then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{k\alpha_1+2k\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{k\alpha_1+(2k-1)\alpha_2}s_1}^\leq \cup \mathbb{O}_{t^{k(\alpha_1+2\alpha_2)}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.9. (1) If $\tilde{w} = t^{k(2\alpha_1+\alpha_2)} s_1 s_2$, with $k \in \mathbb{N}_+$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{(2k-2)\alpha_1 + (k-1)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{(k-1)(2\alpha_1+\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{(2k-1)\alpha_1 + k\alpha_2} s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} = t^{k(2\alpha_1+\alpha_2)} s_2 s_1$, with $k \in \mathbb{N}_+$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{k(2\alpha_1+\alpha_2)} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{k(2\alpha_1+\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{(2k-1)\alpha_1 + k\alpha_2} s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Here we give a proof of 2.8. Symmetrically, we obtain 2.9.

Proof of Proposition 2.8. For (1), we have

$$\begin{aligned} T_{t^{k(\alpha_1+2\alpha_2)} s_2 s_1} &\equiv (v - v^{-1})(T_{t^{k\alpha_1+(2k-1)\alpha_2} s_1} + T_{t^{(k-1)\alpha_1+(2k-2)\alpha_2} s_1}) \\ &\quad + (v - v^{-1})T_{t^{(k-1)\alpha_1+(2k-2)\alpha_2} s_2} + T_{t^{(k-1)(\alpha_1+2\alpha_2)} s_2 s_1} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1})[T_{t^{\alpha_1+\alpha_2} s_1} + \sum_{i=2}^k (T_{t^{(i-1)(\alpha_1+2\alpha_2)} s_1} + T_{t^{i\alpha_1+(2i-1)\alpha_2} s_1})] \\ &\quad + (v - v^{-1}) \sum_{i=1}^{k-1} T_{t^{i(\alpha_1+2\alpha_2)} s_2} + T_{\mathbb{O}_2} \\ &\dots\dots\dots \\ &\equiv T_{\mathbb{O}_2} + (v - v^{-1})^2 \sum_{\lambda \in Q_{(k-1)\alpha_1+(2k-2)\alpha_2}} T_{\mathbb{O}_\lambda} \\ &\quad + (v - v^{-1}) \sum_{\mathbb{O} \in \mathbb{O}_{t^{k\alpha_1+(2k-1)\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{t^{(k-1)(\alpha_1+2\alpha_2)} s_2}^{\leq}} T_{\mathbb{O}}. \end{aligned}$$

Thus (1) is proved. By $T_{t^{k(\alpha_1+2\alpha_2)} s_1 s_2} \equiv (v - v^{-1})T_{t^{k\alpha_1+(2k-1)\alpha_2} s_2} + T_{t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1}$ and (1), (2) is proved. \square

Proposition 2.10. For $k \in \mathbb{N}_+$, we have

$$t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2 \sim t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1,$$

thus $T_{t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2} \equiv T_{t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1}$. If $\tilde{w} = t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2$ or $t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1$ with $k = 1$, then \tilde{w} is already of minimal length in \mathbb{O}_2 . So $f_{\tilde{w}, \mathbb{O}_2} = 1$ and 0 for other \mathbb{O} . When $k \geq 2$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{(k-1)(\alpha_1+2\alpha_2)} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{(k-1)(\alpha_1+2\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{(k-1)(\alpha_1+2\alpha_2)} s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.11. *For $k \in \mathbb{N}_+$, we have*

$$t^{(2k-1)\alpha_1+k\alpha_2} s_1 s_2 \sim t^{(2k-1)\alpha_1+k\alpha_2} s_2 s_1,$$

thus $T_{t^{(2k-1)\alpha_1+k\alpha_2} s_1 s_2} \equiv T_{t^{(2k-1)\alpha_1+k\alpha_2} s_2 s_1}$. If $\tilde{w} = t^{(2k-1)\alpha_1+k\alpha_2} s_1 s_2$ or $t^{(2k-1)\alpha_1+k\alpha_2} s_2 s_1$ with $k = 1$, then \tilde{w} is already of minimal length in \mathbb{O}_2 . So $f_{\tilde{w}, \mathbb{O}_2} = 1$ and 0 for other \mathbb{O} . When $k \geq 2$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{(2k-2)\alpha_1+(k-1)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{(k-1)(2\alpha_1+\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{(k-1)(2\alpha_1+\alpha_2)} s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

The proof of 2.11 is quite similar to that of 2.10, and thus we just give a proof of 2.10.

Proof of Proposition 2.10. Since $t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1 = \tau^{-1} s_0 t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2 s_0 \tau$,

$$t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2 \sim t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1,$$

and hence $T_{t^{k\alpha_1+(2k-1)\alpha_2} s_1 s_2} \equiv T_{t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1}$. When $k = 1$, the statement is obvious. It is sufficient to consider $\tilde{w} = t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1$ where $k \geq 2$. Along with Proposition 2.8 (2),

$$\begin{aligned} T_{t^{k\alpha_1+(2k-1)\alpha_2} s_2 s_1} &\equiv (v - v^{-1}) T_{t^{(k-1)(\alpha_1+2\alpha_2)} s_1} + T_{t^{(k-1)(\alpha_1+2\alpha_2)} s_1 s_2} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1}) \sum_{i=1}^{k-1} (T_{t^{i\alpha_1+(2i-1)\alpha_2} s_1} + T_{t^{i(\alpha_1+2\alpha_2)} s_1}) \\ &\quad + (v - v^{-1}) \sum_{i=1}^{k-1} T_{t^{i(\alpha_1+2\alpha_2)} s_2} + T_{\mathbb{O}_2} \\ &\dots\dots\dots \\ &\equiv T_{\mathbb{O}_2} + (v - v^{-1})^2 \sum_{\lambda \in Q_{(k-1)\alpha_1+(2k-2)\alpha_2}} T_{\mathbb{O}_\lambda} \\ &\quad + (v - v^{-1}) \sum_{\substack{\mathbb{O} \in \mathbb{O}_{t^{(k-1)(\alpha_1+2\alpha_2)} s_1}^{\leq} \\ \cup \mathbb{O}_{t^{(k-1)(\alpha_1+2\alpha_2)} s_2}^{\leq}}} T_{\mathbb{O}}. \end{aligned}$$

The proposition is proved. \square

For $\lambda \in Q \cap P_+$, we set

$$D_\lambda = \{\lambda' \in Q \cap P_+ \mid \lambda' < \lambda - \alpha_1, \lambda' \neq \lambda - i\alpha_1 (i \in \mathbb{N}_+)\} \cap Q_{sh},$$

$$D'_\lambda = \{\lambda' \in Q \cap P_+ \mid \lambda' < \lambda - \alpha_2, \lambda' \neq \lambda - i\alpha_2 (i \in \mathbb{N}_+)\} \cap Q_{sh}.$$

And we set

$$E_\lambda = \{\lambda' \in Q \cap P_+ \mid \lambda' = \lambda - i\alpha_1 (i \in \mathbb{N}_+)\} \cap Q_{sh},$$

$$E'_\lambda = \{\lambda' \in Q \cap P_+ \mid \lambda' = \lambda - i\alpha_2 (i \in \mathbb{N}_+)\} \cap Q_{sh}.$$

Proposition 2.12. (1) If $\tilde{w} = t^\lambda s_1 s_2$ with $\lambda = m\alpha_1 + n\alpha_2$ ($m, n \in \mathbb{Z}$, $2 \leq m \leq n < 2m - 1$), then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda'}, \lambda' \in D_\lambda \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{\lambda-\alpha_1-\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{\lambda-\alpha_1}s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise;} \end{cases}$$

(2) If $\tilde{w} = t^\lambda s_2 s_1$ with $\lambda = m\alpha_1 + n\alpha_2$ ($m, n \in \mathbb{Z}$, $2 \leq m < n < 2m - 1$), then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda'} \text{ with } \lambda' \in D'_\lambda \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{\lambda-\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{\lambda-\alpha_1-\alpha_2}s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.13. (1) If $\tilde{w} = t^\lambda s_2 s_1$ with $\lambda = m\alpha_1 + n\alpha_2$ ($m, n \in \mathbb{Z}$, $2 \leq n \leq m < 2n - 1$), then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda'}, \lambda' \in D'_\lambda \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{\lambda-\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{\lambda-\alpha_1-\alpha_2}s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise;} \end{cases}$$

(2) If $\tilde{w} = t^\lambda s_1 s_2$ with $\lambda = m\alpha_1 + n\alpha_2$ ($m, n \in \mathbb{Z}$, $2 \leq n < m < 2n - 1$), then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda'}, \lambda' \in D_\lambda \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{\lambda-\alpha_1-\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{\lambda-\alpha_1}s_2}^{\leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

To avoid of repetition, we just prove 2.12.

Proof of Proposition 2.12. For (1), let $\lambda = m\alpha_1 + n\alpha_2$

$$\begin{aligned}
T_{t^\lambda s_1 s_2} &\equiv (v - v^{-1})T_{s_0 t^\lambda s_1 s_2} + T_{s_0 t^\lambda s_1 s_2 s_0} \\
&= (v - v^{-1})T_{t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1} + T_{t^{(1-n)\alpha_1 + (2-m)\alpha_2} s_2 s_1} \\
&\equiv (v - v^{-1})T_{\tau^{-1} \cdot t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1} + T_{\tau^{-1} \cdot t^{(1-n)\alpha_1 + (2-m)\alpha_2} s_2 s_1} \\
&= (v - v^{-1})T_{t^{(n-m+1)\alpha_1 + n\alpha_2} s_1 s_2 s_1} + T_{t^{(n-m+1)\alpha_1 + n\alpha_2} s_2 s_1} \\
&\equiv (v - v^{-1})T_{s_1 t^{(n-m+1)\alpha_1 + n\alpha_2} s_1 s_2 s_1 s_1} + (v - v^{-1})T_{s_0 t^{(n-m+1)\alpha_1 + n\alpha_2} s_2 s_1} + T_{s_0 t^{(n-m+1)\alpha_1 + n\alpha_2} s_2 s_1 s_0} \\
&= (v - v^{-1})T_{t^{(m-1)\alpha_1 + n\alpha_2} s_2} + (v - v^{-1})T_{t^{(1-n)\alpha_1 + (m-n)\alpha_2} s_2} + T_{t^{(2-n)\alpha_1 + (m-n)\alpha_2} s_1 s_2} \\
&\equiv (v - v^{-1})T_{t^{(m-1)\alpha_1 + n\alpha_2} s_2} + (v - v^{-1})T_{\tau^{-1} \cdot t^{(1-n)\alpha_1 + (m-n)\alpha_2} s_2} + T_{\tau^{-1} \cdot t^{(2-n)\alpha_1 + (m-n)\alpha_2} s_1 s_2} \\
&= (v - v^{-1})(T_{t^{(m-1)\alpha_1 + n\alpha_2} s_2} + T_{t^{(m-1)\alpha_1 + (n-1)\alpha_2} s_1}) + T_{t^{(m-1)\alpha_1 + (n-1)\alpha_2} s_1 s_2} \\
&\dots\dots\dots \text{(Inductively)} \\
&\equiv (v - v^{-1})[(T_{t^{(m-1)\alpha_1 + n\alpha_2} s_2} + T_{t^{(m-1)\alpha_1 + (n-1)\alpha_2} s_1}) + (T_{t^{(m-2)\alpha_1 + (n-1)\alpha_2} s_2} + T_{t^{(m-2)\alpha_1 + (n-2)\alpha_2} s_1}) + \dots \\
&\quad \dots + (T_{t^{(n-m+1)\alpha_1 + 2(n-m+1)\alpha_2} s_2} + T_{t^{(n-m+1)\alpha_1 + (2(n-m+1)-1)\alpha_2} s_1})] + T_{t^{(n-m+1)\alpha_1 + (2(n-m+1)-1)\alpha_2} s_1 s_2} \\
&\dots\dots\dots \text{(2.10)} \\
&\equiv T_{\mathbb{O}_2} + (v - v^{-1})^2 \sum_{\lambda' \in D_\lambda} T_{\mathbb{O}_{\lambda'}} + (v - v^{-1}) \sum_{\mathbb{O} \in \mathbb{O}_{r^{\lambda-\alpha_1-\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{r^{\lambda-\alpha_1} s_2}^{\prime \leq}} T_{\mathbb{O}}.
\end{aligned}$$

For (2) we combine (1) to obtain

$$\begin{aligned}
T_{t^\lambda s_2 s_1} &\equiv (v - v^{-1})T_{t^{\lambda-\alpha_2} s_1} + T_{t^{\lambda-\alpha_2} s_1 s_2} \\
&\dots\dots\dots \\
&\equiv T_{\mathbb{O}_2} + (v - v^{-1})^2 \sum_{\lambda' \in D'_\lambda} T_{\mathbb{O}_{\lambda'}} + (v - v^{-1}) \sum_{\mathbb{O} \in \mathbb{O}_{r^{\lambda-\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{r^{\lambda-\alpha_1-\alpha_2} s_2}^{\prime \leq}} T_{\mathbb{O}}.
\end{aligned}$$

□

Corollary 2.14. (1) If $\tilde{w} = t^{(2k+1)\alpha_1 + k\alpha_2} s_1 s_2$, $k \in \mathbb{N}$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{2k\alpha_1 + k\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{2k\alpha_1 + k\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{t^{2k\alpha_1 + k\alpha_2} s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} = t^{(2k+1)\alpha_1 + k\alpha_2} s_2 s_1$, $k \in \mathbb{N}$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{2k\alpha_1 + k\alpha_2} \cup E_{(2k+1)\alpha_1 + (k+1)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{(2k+1)\alpha_1 + (k+1)\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{t^{2k\alpha_1 + k\alpha_2} s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_1s_2$, $m, n \in \mathbb{N}$ and $m - 2n > 1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in D'_{m\alpha_1+(m-n)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{m\alpha_1+(m-n)\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{(m-1)\alpha_1+(m-n)\alpha_2}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(4) If $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_2s_1$, $m, n \in \mathbb{N}$ and $m - 2n > 1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in D_{m\alpha_1+(m-n)\alpha_2} \cup E_{m\alpha_1+(m-n)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{m\alpha_1+(m-n)\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{(m-1)\alpha_1+(m-n)\alpha_2}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(5) If $\tilde{w} = t^{m\alpha_1-n\alpha_2}s_1s_2$, $m, n \in \mathbb{N}_+$ and $m - n > 1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in D'_{m\alpha_1+(m+n)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{m\alpha_1+(m+n)\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{(m-1)\alpha_1+(m+n)\alpha_2}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

(6) If $\tilde{w} = t^{m\alpha_1-n\alpha_2}s_2s_1$, $m, n \in \mathbb{N}$ and $m - n > 1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in D_{m\alpha_1+(m+n)\alpha_2} \cup E_{m\alpha_1+(m+n)\alpha_2} \\ (v - v^{-1}), & \mathbb{O} \in \mathbb{O}_{t^{m\alpha_1+(m+n)\alpha_2}s_1}^{\leq} \cup \mathbb{O}_{t^{(m-1)\alpha_1+(m+n)\alpha_2}s_2}^{\prime \leq} \\ 1, & \mathbb{O} = \mathbb{O}_2 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For any \tilde{w} which lies in the Corollary, if we take $\tilde{w}' = s_2\tilde{w}s_2$, then either $T_{\tilde{w}} \equiv (v - v^{-1})T_{s_2\tilde{w}} + T_{\tilde{w}'}$ or $T_{\tilde{w}} \equiv T_{\tilde{w}'}$. At this time, \tilde{w}' appears in the previous propositions. Thus combining the previous propositions, we obtain the Corollary. \square

Proposition 2.15. (1) If $\tilde{w} \in \mathbb{O}_1$ and $\ell(\tilde{w}) = 4k - 1 (k \in \mathbb{N}_+)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (k - i)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i(\alpha_1+\alpha_2)} (1 \leq i \leq k - 1) \\ (k - i)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{i(\alpha_1+\alpha_2)} \cup E'_{i(\alpha_1+\alpha_2)} (3 \leq i \leq k - 1) \\ (k - i)(v - v^{-1})^2, & \mathbb{O} = \mathbb{C}_i \text{ or } \mathbb{C}'_i (1 \leq i \leq k - 1) \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ 1, & \mathbb{O} = \mathbb{O}_1 \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} \in \mathbb{O}_1$ and $\ell(\tilde{w}) = 4k - 3 (k \in \mathbb{N}_+)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(k-1)(\alpha_1+\alpha_2)} \\ (k - 1 - i)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i(\alpha_1+\alpha_2)} (1 \leq i \leq k - 2) \\ (k - 1 - i)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{i(\alpha_1+\alpha_2)} \cup E'_{i(\alpha_1+\alpha_2)} (3 \leq i \leq k - 2) \\ (k - 1 - i)(v - v^{-1})^2, & \mathbb{O} = \mathbb{C}_i \text{ or } \mathbb{C}'_i (1 \leq i \leq k - 2) \\ (k - 1)(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ 1, & \mathbb{O} = \mathbb{O}_1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove (1) and the proof of (2) is similar. Since any element $\tilde{w} \in \mathbb{O}_1$ with $\ell(\tilde{w}) = 4k - 1$ is \sim to an element in $\{\tilde{w} = t^{k\alpha_1} s_1 \mid k \in \mathbb{N}_+\}$, it is enough for us to consider $\tilde{w} = t^{k\alpha_1} s_1$ ($k \in \mathbb{N}_+$).

$$\begin{aligned}
T_{\tilde{w}} &\equiv (v - v^{-1})T_{s_2 \tilde{w}} + T_{s_2 \tilde{w} s_2} \\
&= (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + T_{t^{k(\alpha_1 + \alpha_2)} s_1 s_2 s_1} \\
&\equiv (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{s_0 t^{k(\alpha_1 + \alpha_2)} s_1 s_2 s_1} + T_{s_0 t^{k(\alpha_1 + \alpha_2)} s_1 s_2 s_1 s_0} \\
&= (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{t^{(1-k)(\alpha_1 + \alpha_2)}} + T_{t^{(2-k)(\alpha_1 + \alpha_2)} s_1 s_2 s_1} \\
&\equiv (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{\tau \cdot t^{(1-k)(\alpha_1 + \alpha_2)}} + T_{\tau \cdot t^{(2-k)(\alpha_1 + \alpha_2)} s_1 s_2 s_1} \\
&= (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{t^{(k-1)\alpha_1}} + T_{t^{(k-1)\alpha_1} s_1} \\
&\equiv (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{s_2 t^{(k-1)\alpha_1} s_2} + T_{t^{(k-1)\alpha_1} s_1} \\
&= (v - v^{-1})T_{t^{k(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1})T_{\mathbb{O}_{(k-1)(\alpha_1 + \alpha_2)}} + T_{t^{(k-1)\alpha_1} s_1} \\
&\dots\dots\dots \\
&\equiv (v - v^{-1}) \sum_{i=1}^k T_{t^{i(\alpha_1 + \alpha_2)} s_2 s_1} + (v - v^{-1}) \sum_{i=1}^{k-1} T_{\mathbb{O}_{i(\alpha_1 + \alpha_2)}} + T_{\mathbb{O}_1}.
\end{aligned}$$

Along with Proposition 2.13 (1), we get

$$\begin{aligned}
T_{t^{k\alpha_1} s_1} &\equiv \sum_{i=1}^{k-1} [(k-i)(v - v^{-1})^3 + (v - v^{-1})] T_{\mathbb{O}_{i(\alpha_1 + \alpha_2)}} \\
&\quad + \sum_{i=3}^{k-1} \sum_{\lambda \in E_{i(\alpha_1 + \alpha_2)} \cup E'_{i(\alpha_1 + \alpha_2)}} (k-i)(v - v^{-1})^3 T_{\mathbb{O}_\lambda} \\
&\quad + \sum_{i=1}^{k-1} (k-i)(v - v^{-1})^2 (T_{\mathbb{C}_i} + T_{\mathbb{C}'_i}) + k(v - v^{-1}) T_{\mathbb{O}_2} + T_{\mathbb{O}_1}.
\end{aligned}$$

(1) is proved. □

Proposition 2.16. (1) If $\tilde{w} \in \mathbb{C}_i$ ($i \in \mathbb{N}_+$) and $\ell(\mathbb{C}_i) \leq \ell(\tilde{w}) \leq 6i + 1$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{j\alpha_1 + i\alpha_2} \ (\lfloor \frac{i}{2} \rfloor + 1 \leq j \leq \frac{\ell(\tilde{w})-1}{2} - i) \\ 1, & \mathbb{O} = \mathbb{C}_i \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} \in \mathbb{C}_i (i \in \mathbb{N}_+)$ and $\ell(\tilde{w}) = 6i - 1 + 4k (k \geq 1)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2i\alpha_1 + i\alpha_2} \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ k(v - v^{-1})^2, & \mathbb{O} \in \mathbb{O}_{t^{2i\alpha_1 + i\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{t^{2i\alpha_1 + i\alpha_2} s_2}^{\prime \leq} \setminus \mathbb{C}_i \\ k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{2i\alpha_1 + i\alpha_2} \setminus E_{2i\alpha_1 + i\alpha_2} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{2i\alpha_1 + i\alpha_2} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} \in \mathbb{C}(t^{(2i+j+1)\alpha_1 + (i+j)\alpha_2} s_1) \cup \mathbb{C}'(t^{(2i+j)\alpha_1 + (i+j)\alpha_2} s_2) \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda \lambda \in E_{(2i+j)\alpha_1 + (i+j)\alpha_2} \cup E'_{(2i+j)\alpha_1 + (i+j)\alpha_2} \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i+j)\alpha_1 + (i+j)\alpha_2} \quad (1 \leq j \leq k - 1) \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{C}_i \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\tilde{w} \in \mathbb{C}_i (i \in \mathbb{N}_+)$ and $\ell(\tilde{w}) = 6i + 1 + 4k (k \geq 1)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2i\alpha_1 + i\alpha_2} \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ k(v - v^{-1})^2, & \mathbb{O} \in \mathbb{O}_{t^{2i\alpha_1 + i\alpha_2} s_1}^{\leq} \cup \mathbb{O}_{t^{2i\alpha_1 + i\alpha_2} s_2}^{\prime \leq} \setminus \mathbb{C}_i \\ k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in Q_{2i\alpha_1 + i\alpha_2} \setminus E_{2i\alpha_1 + i\alpha_2} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{2i\alpha_1 + i\alpha_2} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} \in \mathbb{C}(t^{(2i+j+1)\alpha_1 + (i+j)\alpha_2} s_1) \cup \mathbb{C}'(t^{(2i+j)\alpha_1 + (i+j)\alpha_2} s_2) \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda \lambda \in E_{(2i+j)\alpha_1 + (i+j)\alpha_2} \cup E'_{(2i+j)\alpha_1 + (i+j)\alpha_2} \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i+j)\alpha_1 + (i+j)\alpha_2} \quad (1 \leq j \leq k - 1) \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{C}_i \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i+k)\alpha_1 + (i+k)\alpha_2} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.17. (1) If $\tilde{w} \in \mathbb{C}'_i (i \in \mathbb{N}_+)$ and $\ell(\mathbb{C}'_i) \leq \ell(\tilde{w}) \leq 6i + 1$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i\alpha_1 + (\lfloor \frac{i}{2} \rfloor + j)\alpha_2} \quad (1 \leq j \leq \frac{\ell(\tilde{w}) - \ell(\mathbb{C}'_i)}{2}) \\ 1, & \mathbb{O} = \mathbb{C}'_i \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} \in \mathbb{C}'_i (i \in \mathbb{N}_+)$ and $\ell(\tilde{w}) = 6i - 1 + 4k (k \geq 1)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i\alpha_1 + 2i\alpha_2} \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ k(v - v^{-1})^2, & \mathbb{O} \in \mathbb{O}_{t^{i(\alpha_1 + 2\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{i(\alpha_1 + 2\alpha_2)} s_2}^{\prime \leq} \setminus \mathbb{C}'_i \\ k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in \mathcal{Q}_{i\alpha_1 + 2i\alpha_2} \setminus E'_{i\alpha_1 + 2i\alpha_2} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E'_{i\alpha_1 + 2i\alpha_2} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} \in \mathbb{C}(t^{(i+j)\alpha_1 + (2i+j)\alpha_2} s_1) \cup \mathbb{C}'(t^{(i+j)\alpha_1 + (2i+j+1)\alpha_2} s_2) \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{(i+j)\alpha_1 + (2i+j)\alpha_2} \cup E'_{(i+j)\alpha_1 + (2i+j)\alpha_2} \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+j)\alpha_1 + (2i+j)\alpha_2} \quad (1 \leq j \leq k - 1) \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{C}'_i \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\tilde{w} \in \mathbb{C}'_i (i \in \mathbb{N}_+)$ and $\ell(\tilde{w}) = 6i + 1 + 4k (k \geq 1)$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i\alpha_1 + 2i\alpha_2} \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_2 \\ k(v - v^{-1})^2, & \mathbb{O} \in \mathbb{O}_{t^{i(\alpha_1 + 2\alpha_2)} s_1}^{\leq} \cup \mathbb{O}_{t^{i(\alpha_1 + 2\alpha_2)} s_2}^{\prime \leq} \setminus \mathbb{C}'_i \\ k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in \mathcal{Q}_{i\alpha_1 + 2i\alpha_2} \setminus E'_{i\alpha_1 + 2i\alpha_2} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E'_{i\alpha_1 + 2i\alpha_2} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} \in \mathbb{C}(t^{(i+j)\alpha_1 + (2i+j)\alpha_2} s_1) \cup \mathbb{C}'(t^{(i+j)\alpha_1 + (2i+j+1)\alpha_2} s_2) \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_\lambda, \lambda \in E_{(i+j)\alpha_1 + (2i+j)\alpha_2} \cup E'_{(i+j)\alpha_1 + (2i+j)\alpha_2} \\ & (1 \leq j \leq k - 1) \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+j)\alpha_1 + (2i+j)\alpha_2} \quad (1 \leq j \leq k - 1) \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{C}'_i \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+k)\alpha_1 + (2i+k)\alpha_2} \\ 0, & \text{otherwise.} \end{cases}$$

Also, the proofs of 2.16 and 2.17 are similar. We choose 2.16 to prove and leave the other one.

Proof of Proposition 2.16. Since (3) is an easy consequence of (2), we prove (1) and (2). Any element $\tilde{w} \in \mathbb{C}_i$ with $i \in \mathbb{N}_+$ is \sim to an element in $\{t^{k\alpha_1 + (k-i)\alpha_2} s_1 s_2 s_1, t^{k\alpha_1 + i\alpha_2} s_1 \mid k \geq \lfloor \frac{i}{2} \rfloor + 1\} \subset \mathbb{C}_i$. For $\lfloor \frac{i}{2} \rfloor + 1 \leq k \leq 2i$, $t^{k\alpha_1 + i\alpha_2} s_1 = s_2 t^{k\alpha_1 + (k-i)\alpha_2} s_1 s_2 s_1 s_2$ and $\ell(t^{k\alpha_1 + i\alpha_2} s_1) = \ell(t^{k\alpha_1 + (k-i)\alpha_2} s_1 s_2 s_1)$. It is sufficient for us to consider $\tilde{w} \in \{t^{k\alpha_1 + i\alpha_2} s_1 \mid k \geq \lfloor \frac{i}{2} \rfloor + 1\} \sqcup \{t^{k\alpha_1 + (k-i)\alpha_2} s_1 s_2 s_1 \mid k \geq 2i + 1\}$.

(1) For $\tilde{w} \in \mathbb{C}_i$ and $\ell(\mathbb{C}_i) \leq \ell(\tilde{w}) \leq 6i - 1$. We just consider $\tilde{w} = t^{k\alpha_1 + i\alpha_2} s_1$

with $\lfloor \frac{i}{2} \rfloor + 1 \leq k \leq 2i$.

$$\begin{aligned} T_{\tilde{w}} &\equiv (v - v^{-1})T_{\mathbb{O}_{(k-1)\alpha_1 + i\alpha_2}} + T_{t^{(k-1)\alpha_1 + i\alpha_2}s_1} \\ &\quad \dots \\ &\equiv (v - v^{-1}) \sum_{j=\lfloor \frac{i}{2} \rfloor + 1}^{k-1} T_{\mathbb{O}_{j\alpha_1 + i\alpha_2}} + T_{\mathbb{C}_i}. \end{aligned}$$

If $\ell(\tilde{w}) = 6i + 1$, we can take $\tilde{w} = t^{(2i+1)\alpha_1 + (i+1)\alpha_2}s_1s_2s_1$. Then

$$T_{t^{(2i+1)\alpha_1 + (i+1)\alpha_2}s_1s_2s_1} \equiv (v - v^{-1}) \sum_{\lambda \in E_{2i\alpha_1 + i\alpha_2} \cup \{2i\alpha_1 + i\alpha_2\}} T_{\mathbb{O}_\lambda} + T_{\mathbb{C}_i}.$$

(2) We consider the case $\tilde{w} = t^{(2i+k)\alpha_1 + i\alpha_2}s_1$, with $k \geq 1$

$$\begin{aligned} T_{\tilde{w}} &\equiv (v - v^{-1})T_{t^{(2i+k)\alpha_1 + (k+i)\alpha_2}s_2s_1} + T_{t^{(2i+k)\alpha_1 + (k+i)\alpha_2}s_1s_2s_1} \\ &\equiv (v - v^{-1})T_{t^{(2i+k)\alpha_1 + (k+i)\alpha_2}s_2s_1} + (v - v^{-1})T_{\mathbb{O}_{(2i+k-1)\alpha_1 + (k+i-1)\alpha_2}} + T_{t^{(2i+k-1)\alpha_1 + i\alpha_2}s_1} \\ &\quad \dots \dots \dots \\ &\equiv (v - v^{-1}) \sum_{j=1}^k T_{t^{(2i+j)\alpha_1 + (j+i)\alpha_2}s_2s_1} + (v - v^{-1}) \sum_{j=0}^{k-1} T_{\mathbb{O}_{(2i+j)\alpha_1 + (j+i)\alpha_2}} \\ &\quad + (v - v^{-1}) \sum_{j=\lfloor \frac{i}{2} \rfloor + 1}^{2i-1} T_{\mathbb{O}_{j\alpha_1 + i\alpha_2}} + T_{\mathbb{C}_i}. \end{aligned}$$

Together with Proposition 2.13 (1), then (2) is proved. \square

2.2. Class polynomials for elements in $W_a\tau$. We set

$$\begin{aligned} \mathbb{I}\tau &= \{t^{m\alpha_1 + n\alpha_2}w\tau \mid m \in \mathbb{N}_+, n \in \mathbb{Z}, 1 - m \leq n \leq 2m - 1, w \in W\} \sqcup \{\tau, s_2\tau\} \\ &\sqcup \{t^{k\alpha_1 + 2k\alpha_2}\tau, t^{k\alpha_1 + 2k\alpha_2}s_2\tau, t^{k\alpha_1 + 2k\alpha_2}s_2s_1\tau, t^{k(\alpha_1 - \alpha_2)}\tau, t^{k(\alpha_1 - \alpha_2)}s_1\tau, t^{k(\alpha_1 - \alpha_2)}s_2\tau \mid k \in \mathbb{N}_+\}. \end{aligned}$$

Since $W_a\tau = \mathbb{I}\tau \cup \tau \cdot (\mathbb{I}\tau) \cup \tau^2 \cdot (\mathbb{I}\tau)$ and any element $\tilde{w} \in W_a\tau$ is \sim to an element in $\mathbb{I}\tau$, it is sufficient to consider elements in $\mathbb{I}\tau$. Now we are going to classify all \tilde{W} -conjugacy classes in $W_a\tau$.

Lemma 2.18. *Let $\mathbb{O}_{id,\tau}$ be the set of \tilde{W} -conjugacy class in $W_a\tau$ with minimal length 0. Then*

$$\mathbb{O}_{id,\tau} = \{t^{m\alpha_1 + n\alpha_2}\tau, t^{m\alpha_1 + n\alpha_2}s_1s_2\tau \mid m, n \in \mathbb{Z}\}.$$

Lemma 2.19. *For any $\lambda \in P_+ \cap Q$ and $\lambda \neq 2m\alpha_1 + m\alpha_2$ ($m \in \mathbb{N}$) i.e. $\lambda = m\alpha_1 + n\alpha_2$ where $1 \leq m \leq n \leq 2m$ or $1 \leq n < m \leq 2n - 1$, we set $\mathbb{O}_{\lambda,\tau} = \tilde{W} \cdot (t^\lambda s_2 s_1 \tau)$. If $n = 2m$ or $m = 2n - 1$ where $m, n \in \mathbb{N}_+$, then*

$$\mathbb{O}_{\lambda,\tau} = \{t^{m\alpha_1 + n\alpha_2}s_2s_1\tau, t^{(1-n)\alpha_1 + (m+1-n)\alpha_2}s_2s_1\tau, t^{(n-m)\alpha_1 + (1-m)\alpha_2}s_2s_1\tau\}.$$

If $2 \leq m \leq n \leq 2m - 1$ or $1 \leq n < m < 2n - 1$, then

$$\begin{aligned} \mathbb{O}_{\lambda,\tau} &= \{t^{m\alpha_1 + n\alpha_2}s_2s_1\tau, t^{(1-n)\alpha_1 + (m+1-n)\alpha_2}s_2s_1\tau, t^{(n-m)\alpha_1 + (1-m)\alpha_2}s_2s_1\tau\} \\ &\sqcup \{t^{(n-m)\alpha_1 + n\alpha_2}s_2s_1\tau, t^{(1-n)\alpha_1 + (1-m)\alpha_2}s_2s_1\tau, t^{m\alpha_1 + (m+1-n)\alpha_2}s_2s_1\tau\} \end{aligned}$$

and $\ell(\mathbb{O}_{\lambda,\tau}) = \ell(t^\lambda s_2 s_1 \tau)$.

Lemma 2.20. For $i \in \mathbb{Z}$, let $\mathbb{O}_{i,\tau} = \widetilde{W} \cdot (t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau)$. Then $\mathbb{O}_{i,\tau}$ is the set of \widetilde{W} -conjugacy class in $W_a \tau$ with

$$\ell(\mathbb{O}_{i,\tau}) = \ell(t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau).$$

Moreover, $\mathbb{O}_{i,\tau} = \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau, t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau, t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\}$.

Since the approach of proving the above lemmas 2.18, 2.19 and 2.20 are quite similar, here we give a proof of 2.20 as an example.

Proof of Lemma 2.20. (1) We set

$$A = \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau, t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau, t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\},$$

then $\widetilde{W} \cdot A \subset A$. Since

$$\begin{aligned} A &= \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \{t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \{t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau \mid k \in \mathbb{Z}\} \\ &= \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \tau \cdot \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \tau^{-1} \cdot \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \in \mathbb{Z}\} \\ &= \{t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau \mid k \in \mathbb{Z}\} \sqcup \tau \cdot \{t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau \mid k \in \mathbb{Z}\} \sqcup \tau^{-1} \cdot \{t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau \mid k \in \mathbb{Z}\} \\ &= \{t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \tau \cdot \{t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\} \sqcup \tau^{-1} \cdot \{t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau \mid k \in \mathbb{Z}\}, \\ s_0 \cdot (t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau) &= t^{(1-i)\alpha_1 + (1-k)\alpha_2} s_1 \tau, s_1 \cdot (t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau) = t^{(i-k)\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau, s_2 \cdot \\ (t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau) &= t^{(k-1)\alpha_1 + (k-i)\alpha_2} s_2 \tau, s_0 \cdot (t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau) = t^{(1-k)\alpha_1 + (2-i-k)\alpha_2} s_2 \tau, s_1 \cdot \\ (t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau) &= t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau, s_2 \cdot (t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau) = t^{(k+i)\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau, \\ s_0 \cdot (t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau) &= t^{(1-k)\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau, s_1 \cdot (t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau) = t^{(k+i-1)\alpha_1 + k\alpha_2} s_2 \tau, s_2 \cdot \\ (t^{(1-i)\alpha_1 + k\alpha_2} s_1 \tau) &= t^{(1-i)\alpha_1 - (k+i)\alpha_2} s_1 \tau, \text{ thus (1) is proved.} \end{aligned}$$

(2) For any $\tilde{w} \in \widetilde{W}$, by direct calculation we have

$$\ell(t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau) \leq \ell(\tilde{w} t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau \tilde{w}^{-1}).$$

For any $\tilde{w} \in A$, \tilde{w} is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. We use induction on length. If for all $\tilde{w}' \in A$ with $\ell(\tilde{w}') < \ell(\tilde{w})$, then \tilde{w}' is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. From the proof of (1), it is sufficient for us to consider that $\tilde{w} \in \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \in \mathbb{Z}\}$. Now we take $\tilde{w} = t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau$. (a) For $i > 0$ and i is odd. If $k \geq 2i$, then we take $\tilde{w}_1 = s_0 \cdot (\tilde{w})$. Then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus by induction \tilde{w} is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. If $\lfloor \frac{i}{2} \rfloor + 2 \leq k \leq 2i - 1$, then we take $\tilde{w}_1 = s_2 s_0 \cdot (\tilde{w})$. Then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus by induction \tilde{w} is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. If $k = \lfloor \frac{i}{2} \rfloor + 1$, then \tilde{w} is already of minimal length. If $k \leq \lfloor \frac{i}{2} \rfloor$, then we take $\tilde{w}_1 = s_1 \cdot (\tilde{w})$. Then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus by induction \tilde{w} is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. (b) For $i > 0$ and i is even. If $k \geq 2i$ or $\lfloor \frac{i}{2} \rfloor + 2 \leq k \leq 2i - 1$ or $k \leq \lfloor \frac{i}{2} \rfloor - 1$ then \tilde{w}_1 is as before, and thus \tilde{w} is \widetilde{W} -conjugate to $t^{\lfloor \frac{i}{2} \rfloor + 1} \alpha_1 + i \alpha_2 s_1 s_2 s_1 \tau$. If $i \leq 0$, the argument is similar we omit it here. By (1) and (2), the lemma is proved. \square

Theorem 2.21. The \widetilde{W} -conjugacy classes in $W_a \tau$ are: $\mathbb{O}_{id,\tau}$, $\mathbb{O}_{\lambda,\tau}$, and $\mathbb{O}_{i,\tau}$ with $i \in \mathbb{Z}$, $\lambda \in P_+ \cap Q$ and $\lambda \neq 2m\alpha_1 + m\alpha_2$ ($m \in \mathbb{N}$).

Proof. Follows directly from Lemmas 2.18, 2.19 and 2.20. \square

Proposition 2.22. *For any $\tilde{w} \in \mathbb{O}_{\lambda, \tau}$ where $\lambda \in P_+ \cap Q$ and $\lambda \neq 2m\alpha_1 + m\alpha_2$ ($m \in \mathbb{N}$). Then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} 1, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Following the definition of class polynomials, the proposition is obvious. \square

For $\lambda = m\alpha_1 + n\alpha_2 \in P_+ \cap Q_{sh}$, we set

$$E_{m\alpha_1 + n\alpha_2, \tau} = \{\lambda' = k\alpha_1 + n\alpha_2 \mid \lfloor \frac{n}{2} \rfloor + 1 \leq k \leq m-1\},$$

$$E'_{m\alpha_1 + n\alpha_2, \tau} = \{\lambda' = m\alpha_1 + k\alpha_2 \mid \lfloor \frac{m+1}{2} \rfloor + 1 \leq k \leq n\}.$$

Proposition 2.23. (1) *Let $\lambda = m\alpha_1 + n\alpha_2 \in P_+ \cap Q_{sh}$, and $\tilde{w} = t^\lambda s_1 s_2 s_1 \tau$. Then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda', \tau}, \text{ with } \lambda' \in E_{m\alpha_1 + n\alpha_2, \tau} \\ 1, & \mathbb{O} = \mathbb{O}_{n, \tau} \\ 0, & \text{otherwise} \end{cases}$$

(2) *Let $\lambda = m\alpha_1 + n\alpha_2 \in P_+ \cap Q_{sh}$ with $n \neq 2m$, and $\tilde{w} = t^\lambda s_1 \tau$. Then*

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda', \tau}, \text{ with } \lambda' \in E'_{m\alpha_1 + n\alpha_2, \tau} \\ 1, & \mathbb{O} = \mathbb{O}_{1-m, \tau} \\ 0, & \text{otherwise} \end{cases}$$

Proof. (1): If $\lambda = k\alpha_1 + (2k-1)\alpha_2$ or $\lambda = (k+1)\alpha_1 + 2k\alpha_2$ where $k \in \mathbb{N}_+$, then $\tilde{w} = t^\lambda s_1 s_2 s_1 \tau$ is already of minimal length in its \tilde{W} -conjugacy class. Thus it is sufficient to consider those $\lambda = m\alpha_1 + n\alpha_2 \in P_+ \cap Q_{sh}$ with $n \leq 2m-3$, in this case we have

$$\begin{aligned} T_{t^{m\alpha_1 + n\alpha_2} s_1 s_2 s_1 \tau} &\equiv T_{s_0 t^{m\alpha_1 + n\alpha_2} s_1 s_2 s_1 \tau s_0} = T_{t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1 \tau} \\ &\equiv (v - v^{-1}) T_{s_2 t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1 \tau} + T_{s_2 t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1 \tau s_2} \\ &\equiv (v - v^{-1}) T_{\tau^{-1} t^{(1-n)\alpha_1 + (m-n)\alpha_2} s_2 s_1 \tau} + T_{\tau^{-1} t^{(1-n)\alpha_1 + (m-n-1)\alpha_2} s_1 \tau} \\ &\equiv (v - v^{-1}) T_{t^{(m-1)\alpha_1 + n\alpha_2} s_2 s_1 \tau} + T_{t^{(m-1)\alpha_1 + n\alpha_2} s_1 s_2 s_1 \tau} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1}) \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{m-1} T_{\mathbb{O}_{i\alpha_1 + n\alpha_2, \tau}} + T_{\mathbb{O}_{n, \tau}}. \end{aligned}$$

(2): If $\lambda = (2k-1)\alpha_1 + k\alpha_2$, then $\tilde{w} = t^\lambda s_1 \tau$ is a minimal length element in its \tilde{W} -conjugacy class. Thus it is sufficient to consider $\lambda = m\alpha_1 + n\alpha_2 \in P_+ \cap Q_{sh}$ with $m \leq 2n-2$, thus by a similar argument we have

$$\begin{aligned} T_{t^{m\alpha_1 + n\alpha_2} s_1 \tau} &\equiv T_{s_0 t^{m\alpha_1 + n\alpha_2} s_1 \tau s_0} = T_{t^{(1-n)\alpha_1 + (1-m)\alpha_2} s_1 s_2 s_1 \tau} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1}) \sum_{i=\lfloor \frac{m+1}{2} \rfloor + 1}^n T_{\mathbb{O}_{m\alpha_1 + i\alpha_2, \tau}} + T_{\mathbb{O}_{1-m, \tau}}. \end{aligned}$$

Thus (2) is proved. \square

Note: $\tilde{w} = t^{2k\alpha_1+k\alpha_2}s_1\tau$ for $k \in \mathbb{N}_+$ is a minimal length element of $\mathbb{O}_{1-2k,\tau}$.

Proposition 2.24. *If $\tilde{w} = t^{k\alpha_1+2k\alpha_2}\tau$ with $k \in \mathbb{N}$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^k E'_{j\alpha_1+(2j-1)\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -k+1 \leq i \leq 2k \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise} \end{cases}$$

Proof. If $k = 0$, there is nothing to prove. If $k = 1$, then

$$T_{t^{\alpha_1+2\alpha_2}\tau} \equiv (v - v^{-1})(T_{\mathbb{O}_{2,\tau}} + T_{\mathbb{O}_{1,\tau}} + T_{\mathbb{O}_{0,\tau}}) + T_{\mathbb{O}_{id,\tau}}.$$

If $k \geq 2$, then

$$\begin{aligned} T_{t^{k\alpha_1+2k\alpha_2}\tau} &\equiv (v - v^{-1})T_{t^{k\alpha_1+(1-k)\alpha_2}s_2\tau} + T_{t^{k\alpha_1+(1-k)\alpha_2}\tau} \\ &\equiv (v - v^{-1})T_{s_2t^{k\alpha_1+(1-k)\alpha_2}s_2\tau} + (v - v^{-1})T_{s_0t^{k\alpha_1+(1-k)\alpha_2}\tau} + T_{s_0t^{k\alpha_1+(1-k)\alpha_2}\tau s_0} \\ &\equiv (v - v^{-1})T_{\mathbb{O}_{2k,\tau}} + (v - v^{-1})T_{t^{k\alpha_1+(1-k)\alpha_2}s_1s_2s_1\tau} + T_{t^{k\alpha_1+(1-k)\alpha_2}s_1s_2\tau} \\ &\equiv (v - v^{-1})T_{\mathbb{O}_{2k,\tau}} + (v - v^{-1})T_{\tau \cdot (t^{k\alpha_1+(1-k)\alpha_2}s_1s_2s_1\tau)} + T_{\tau \cdot (t^{k\alpha_1+(1-k)\alpha_2}s_1s_2\tau)} \\ &\equiv (v - v^{-1})T_{\mathbb{O}_{2k,\tau}} + (v - v^{-1})T_{t^{k\alpha_1+(2k-1)\alpha_2}s_1\tau} \\ &\quad + (v - v^{-1})T_{s_0t^{k\alpha_1+(2k-1)\alpha_2}s_1s_2\tau} + T_{s_0t^{k\alpha_1+(2k-1)\alpha_2}s_1s_2\tau s_0} \\ &\equiv (v - v^{-1})T_{\mathbb{O}_{2k,\tau}} + (v - v^{-1})T_{t^{k\alpha_1+(2k-1)\alpha_2}s_1\tau} \\ &\quad + (v - v^{-1})T_{\tau^{-1} \cdot (t^{(2-2k)\alpha_1+(1-k)\alpha_2}s_1\tau)} + T_{\tau^{-1} \cdot (t^{(2-2k)\alpha_1+(1-k)\alpha_2}\tau)} \\ &= (v - v^{-1})(T_{\mathbb{O}_{2k,\tau}} + T_{\mathbb{O}_{2k-1,\tau}}) + (v - v^{-1})T_{t^{k\alpha_1+(2k-1)\alpha_2}s_1\tau} + T_{t^{(k-1)\alpha_1+2(k-1)\alpha_2}\tau} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1})(T_{\mathbb{O}_{2k,\tau}} + T_{\mathbb{O}_{2k-1,\tau}} + T_{\mathbb{O}_{2k-2,\tau}} + T_{\mathbb{O}_{2k-3,\tau}} + \dots + T_{\mathbb{O}_{4,\tau}} + T_{\mathbb{O}_{3,\tau}}) \\ &\quad + (v - v^{-1})(T_{t^{k\alpha_1+(2k-1)\alpha_2}s_1\tau} + T_{t^{(k-1)\alpha_1+(2k-3)\alpha_2}s_1\tau} + \dots + T_{t^{2\alpha_1+3\alpha_2}s_1\tau}) + T_{t^{\alpha_1+2\alpha_2}\tau} \\ &\equiv (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^k E'_{j\alpha_1+(2j-1)\alpha_2,\tau}} T_{\mathbb{O}_{\lambda,\tau}} + (v - v^{-1}) \sum_{i=-k+1}^{2k} T_{\mathbb{O}_{i,\tau}} + T_{\mathbb{O}_{id,\tau}}. \end{aligned}$$

The proposition is proved. \square

Proposition 2.25. *If $\tilde{w} = t^{k\alpha_1+(2k-1)\alpha_2}s_1s_2\tau$ with $k \in \mathbb{N}_+$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{k-1} E'_{j\alpha_1+(2j-1)\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -k+2 \leq i \leq 2k-1 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.26. *If $\tilde{w} = t^{k\alpha_1+(2k-1)\alpha_2}\tau$ with $k \in \mathbb{N}_+$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^k E'_{j\alpha_1+(2j-1)\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -k+1 \leq i \leq 2k-1 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise} \end{cases}$$

The proofs of 2.25 and 2.26 are just like that of 2.24. To avoid of tedious repetitions, we omit them.

Remark 2.27. For $k \in \mathbb{N}_+$, since $t^{(k+1)\alpha_1+2k\alpha_2} s_1 s_2 \tau = s_2(\tau^{-1} \cdot t^{k\alpha_1+2k\alpha_2} \tau) s_2$, we have $t^{k\alpha_1+2k\alpha_2} \tau \sim t^{(k+1)\alpha_1+2k\alpha_2} s_1 s_2 \tau$ and thus $T_{t^{k\alpha_1+2k\alpha_2} \tau} \equiv T_{t^{(k+1)\alpha_1+2k\alpha_2} s_1 s_2 \tau}$.

Proposition 2.28. If $\tilde{w} = t^{2k\alpha_1+k\alpha_2} s_1 s_2 \tau$ where $k \in \mathbb{N}_+$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^k E_{(2j-1)\alpha_1+j\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 2 - 2k \leq i \leq k \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.29. If $\tilde{w} = t^{2k\alpha_1+k\alpha_2} \tau$ where $k \in \mathbb{N}_+$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^k E_{(2j-1)\alpha_1+j\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 1 - 2k \leq i \leq k \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.30. If $\tilde{w} = t^{(2k-1)\alpha_1+k\alpha_2} s_1 s_2 \tau$ where $k \in \mathbb{N}_{\geq 2}$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^k E_{(2j-1)\alpha_1+j\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 3 - 2k \leq i \leq k \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise} \end{cases}$$

We prove 2.28, and since the proofs of 2.29 and 2.30 are similar, we omit them.

Proof of Proposition 2.28. If $k = 1$, then $T_{t^{2\alpha_1+\alpha_2} s_1 s_2 \tau} \equiv (v - v^{-1})(T_{\mathbb{O}_{0, \tau}} + T_{\mathbb{O}_{1, \tau}}) + T_{\mathbb{O}_{id, \tau}}$. If for $k \geq 2$, then

$$\begin{aligned} T_{t^{2k\alpha_1+k\alpha_2} s_1 s_2 \tau} &\equiv T_{s_0 t^{2k\alpha_1+k\alpha_2} s_1 s_2 \tau s_0} = T_{t^{(1-k)\alpha_1+(1-2k)\alpha_2} \tau} \\ &\equiv (v - v^{-1}) T_{t^{(1-k)\alpha_1+k\alpha_2} s_2 \tau} + T_{t^{(2-k)\alpha_1+k\alpha_2} s_1 s_2 \tau} \\ &\equiv (v - v^{-1}) T_{\tau^{-1} \cdot t^{(1-k)\alpha_1+k\alpha_2} s_2 \tau} + T_{\tau^{-1} \cdot t^{(2-k)\alpha_1+k\alpha_2} s_1 s_2 \tau} \\ &\equiv (v - v^{-1}) T_{t^{(2k-1)\alpha_1+(k-1)\alpha_2} s_1 \tau} + (v - v^{-1}) T_{s_2 t^{(2k-1)\alpha_1+(k-1)\alpha_2} s_1 s_2 \tau} + T_{s_2 t^{(2k-1)\alpha_1+(k-1)\alpha_2} s_1 s_2 \tau s_2} \\ &\dots \dots \dots \\ &\equiv (v - v^{-1}) (T_{t^{(2k-1)\alpha_1+(k-1)\alpha_2} s_1 \tau} + T_{t^{(2k-1)\alpha_1+k\alpha_2} s_1 s_2 s_1 \tau} + T_{t^{(2k-2)\alpha_1+(k-1)\alpha_2} s_1 \tau}) + T_{t^{(2k-2)\alpha_1+(k-1)\alpha_2} s_1 s_2 \tau} \\ &\dots \dots \dots \\ &\equiv (v - v^{-1}) \sum_{i=-2(k-1)}^0 T_{\mathbb{O}_{i, \tau}} + (v - v^{-1}) \sum_{i=1}^k T_{t^{(2i-1)\alpha_1+i\alpha_2} s_1 s_2 s_1 \tau} + T_{\mathbb{O}_{id, \tau}} \\ &\equiv (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^k E_{(2j-1)\alpha_1+j\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + (v - v^{-1}) \sum_{i=-2(k-1)}^k T_{\mathbb{O}_{i, \tau}} + T_{\mathbb{O}_{id, \tau}}. \end{aligned}$$

□

Remark 2.31. For $k \in \mathbb{N}_{\geq 2}$, since $t^{(2k-1)\alpha_1+k\alpha_2}\tau = s_1(\tau \cdot (t^{2k\alpha_1+k\alpha_2}s_1s_2\tau))s_1$ and $\ell(t^{2k\alpha_1+k\alpha_2}s_1s_2\tau) = \ell(t^{(2k-1)\alpha_1+k\alpha_2}\tau)$, thus $T_{t^{2k\alpha_1+k\alpha_2}s_1s_2\tau} \equiv T_{t^{(2k-1)\alpha_1+k\alpha_2}\tau}$.

Proposition 2.32. Let $\tilde{w} = t^{m\alpha_1+n\alpha_2}\tau$ where $m, n \in \mathbb{N}_+$, $m\alpha_1+n\alpha_2 \in P_+ \cap Q_{sh}$, $m \neq 2n-1$ and $n \neq 2m-1$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^{\lfloor \frac{n+1}{2} \rfloor} E'_{j\alpha_1+(2j-1)\alpha_2, \tau} \sqcup \sqcup_{j=\lfloor \frac{n+1}{2} \rfloor+1}^m E'_{j\alpha_1+n\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 1 - m \leq i \leq n \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.33. Let $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_1s_2\tau$ where $m, n \in \mathbb{N}_+$, $\lambda \in P_+ \cap Q_{sh}$, $m \neq 2n-1$ and $n \neq 2m-1, n \neq 2m-2$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^{\lfloor \frac{m+1}{2} \rfloor} E_{(2j-1)\alpha_1+j\alpha_2, \tau} \sqcup \sqcup_{j=\lfloor \frac{m+1}{2} \rfloor+1}^n E_{m\alpha_1+j\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 2 - m \leq i \leq n \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise} \end{cases}$$

We prove 2.32, and the proof of 2.33 is similar which will be omitted.

Proof of Proposition 2.32. If n is odd, then

$$\begin{aligned} T_{t^{m\alpha_1+n\alpha_2}\tau} &\equiv (v - v^{-1})T_{t^{(1-n)\alpha_1+(1-m)\alpha_2}s_1s_2s_1\tau} + T_{t^{(1-n)\alpha_1+(1-m)\alpha_2}s_1s_2\tau} \\ &\equiv (v - v^{-1})T_{\tau^{-1} \cdot (s_2t^{(1-n)\alpha_1+(1-m)\alpha_2}s_1s_2s_1\tau s_2)} + T_{\tau^{-1} \cdot (s_2t^{(1-n)\alpha_1+(1-m)\alpha_2}s_1s_2\tau s_2)} \\ &= (v - v^{-1})T_{t^{m\alpha_1+n\alpha_2}s_1\tau} + T_{t^{(m-1)\alpha_1+n\alpha_2}\tau} \\ &\dots\dots\dots \\ &\equiv (v - v^{-1}) \sum_{j=\frac{n+1}{2}+1}^m T_{t^{j\alpha_1+n\alpha_2}s_1\tau} + T_{t^{\frac{n+1}{2}\alpha_1+n\alpha_2}\tau} \\ &\equiv (v - v^{-1}) \sum_{j=1-m}^{-\frac{n+1}{2}} T_{\mathbb{O}_{j, \tau}} + (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=\frac{n+1}{2}+1}^m E'_{j\alpha_1+n\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} \\ &\quad + (v - v^{-1}) \sum_{j=1-\frac{n+1}{2}}^n T_{\mathbb{O}_{j, \tau}} + (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^{\frac{n+1}{2}} E'_{j\alpha_1+(2j-1)\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + T_{\mathbb{O}_{id, \tau}} \\ &= (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^{\frac{n+1}{2}} E'_{j\alpha_1+(2j-1)\alpha_2, \tau} \sqcup \sqcup_{j=\frac{n+1}{2}+1}^m E'_{j\alpha_1+n\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + (v - v^{-1}) \sum_{i=1-m}^n T_{\mathbb{O}_{i, \tau}} + T_{\mathbb{O}_{id, \tau}}. \end{aligned}$$

If n is even, then by a similar argument

$$\begin{aligned} T_{t^{m\alpha_1+n\alpha_2}\tau} &\equiv (v - v^{-1}) \sum_{i=\frac{n}{2}+1}^m T_{t^{i\alpha_1+n\alpha_2}s_1\tau} + T_{t^{\frac{n}{2}\alpha_1+n\alpha_2}\tau} \\ &\equiv (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^{\frac{n}{2}} E'_{j\alpha_1+(2j-1)\alpha_2,\tau} \sqcup \sqcup_{j=\frac{n}{2}+1}^m E'_{j\alpha_1+n\alpha_2,\tau}} T_{\mathbb{O}_{\lambda,\tau}} + (v - v^{-1}) \sum_{i=1-m}^n T_{\mathbb{O}_{i,\tau}} + T_{\mathbb{O}_{id,\tau}}. \end{aligned}$$

Combine them together, the proposition is proved. \square

Remark 2.34. If $\tilde{w} = t^{m\alpha_1+n\alpha_2}\tau$ where $n \geq 1$, $m \geq 2n+1$ or $n \leq 0$, $m+n \geq 1$, and we set $\tilde{w}_1 = s_2\tilde{w}s_2 = t^{(m+1)\alpha_1+(m-n)\alpha_2}s_1s_2\tau$. Then $\ell(\tilde{w}_1) = \ell(\tilde{w})$, $\tilde{w} \sim \tilde{w}_1$ and hence $T_{\tilde{w}} \equiv T_{\tilde{w}_1}$. We know that \tilde{w}_1 is contained in the situation we considered above.

Corollary 2.35. (1) If $\tilde{w} = t^{(2n+1)\alpha_1+n\alpha_2}\tau$ where $n \in \mathbb{N}_+$. Then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{n+1} E_{(2j-1)\alpha_1+j\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -2n \leq i \leq n+1 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} = t^{(2n+2)\alpha_1+n\alpha_2}\tau$ where $n \in \mathbb{N}_+$. Then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{n+2} E_{(2j-1)\alpha_1+j\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -1-2n \leq i \leq n+2 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\tilde{w} = t^{m\alpha_1+n\alpha_2}\tau$ where $n \geq 1$, $m \geq 2n+3$ or $n \leq 0$, $m+n \geq 1$. Then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{\lfloor \frac{m+2}{2} \rfloor} E_{(2j-1)\alpha_1+j\alpha_2,\tau} \sqcup \sqcup_{j=\lfloor \frac{m+2}{2} \rfloor+1}^{m-n} E_{(m+1)\alpha_1+j\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } 1-m \leq i \leq m-n \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.36. If $\tilde{w} = t^{m\alpha_1+n\alpha_2}s_1s_2\tau$ where $n \geq 1$, $m \geq 2n+1$ or $n \leq 0$, $m+n \geq 2$, then $T_{t^{m\alpha_1+n\alpha_2}s_1s_2\tau} \equiv (v - v^{-1})T_{t^{m\alpha_1+(m-n)\alpha_2}s_1s_2s_1\tau} + T_{t^{(m-1)\alpha_1+(m-n-1)\alpha_2}\tau}$.

Corollary 2.37. (1) If $\tilde{w} = t^{(2k+1)\alpha_1+k\alpha_2}s_1s_2\tau$ where $k \in \mathbb{N}_+$, then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{k+1} E_{(2j-1)\alpha_1+j\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } 1-2k \leq i \leq k+1 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\tilde{w} = t^{(2k+2)\alpha_1+k\alpha_2}s_1s_2\tau$ where $k \in \mathbb{N}_+$, then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda,\tau}, \text{ where } \lambda \in \sqcup_{j=2}^{k+1} E_{(2j-1)\alpha_1+j\alpha_2,\tau} \sqcup E_{(2k+2)\alpha_1+(k+2)\alpha_2,\tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i,\tau}, \text{ where } -2k \leq i \leq k+2 \\ 1, & \mathbb{O} = \mathbb{O}_{id,\tau} \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\tilde{w} = t^{(k+2)\alpha_1 - k\alpha_2} s_1 s_2 \tau$ where $k \in \mathbb{N}$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^{k+1} E'_{j\alpha_1 + (2j-1)\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } -k \leq i \leq 2k + 2 \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise.} \end{cases}$$

(4) If $\tilde{w} = t^{m\alpha_1 + n\alpha_2} s_1 s_2 \tau$ where $n \in \mathbb{N}_+$, $m \geq 2n + 3$ or $n \in \mathbb{Z}_{\leq 0}$, $m + n \geq 3$ then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where} \\ & \lambda \in \sqcup_{j=2}^{\lfloor \frac{m-n}{2} \rfloor} E'_{j\alpha_1 + (2j-1)\alpha_2, \tau} \sqcup \sqcup_{j=\lfloor \frac{m-n}{2} \rfloor + 1}^{m-1} E'_{j\alpha_1 + (m-n-1)\alpha_2, \tau} \sqcup E_{m\alpha_1 + (m-n)\alpha_2, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{i, \tau}, \text{ where } 2 - m \leq i \leq m - n \\ 1, & \mathbb{O} = \mathbb{O}_{id, \tau} \\ 0, & \text{otherwise.} \end{cases}$$

Proofs of Corollary 2.35 and 2.37. We choose 2.35 (1) and 2.37 (4) to prove. For 2.35 (1), it follows directly from

$$\begin{aligned} T_{t^{(2n+1)\alpha_1 + n\alpha_2} \tau} &\equiv T_{t^{(2n+2)\alpha_1 + (n+1)\alpha_2} s_1 s_2 \tau} \\ &\equiv (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^{n+1} E_{(2j-1)\alpha_1 + j\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + (v - v^{-1}) \sum_{i=-2n}^{n+1} T_{\mathbb{O}_{i, \tau}} + T_{\mathbb{O}_{id, \tau}}. \end{aligned}$$

And for 2.37 (4), we obtain the class polynomials form

$$\begin{aligned} T_{t^{m\alpha_1 + n\alpha_2} s_1 s_2 \tau} &\equiv (v - v^{-1}) T_{t^{m\alpha_1 + (m-n)\alpha_2} s_1 s_2 s_1 \tau} + T_{t^{(m-1)\alpha_1 + (m-n-1)\alpha_2} \tau} \\ &\equiv (v - v^{-1})^2 \sum_{\lambda \in \sqcup_{j=2}^{\lfloor \frac{m-n}{2} \rfloor} E'_{j\alpha_1 + (2j-1)\alpha_2, \tau} \sqcup \sqcup_{j=\lfloor \frac{m-n}{2} \rfloor + 1}^{m-1} E'_{j\alpha_1 + (m-n-1)\alpha_2, \tau} \sqcup E_{m\alpha_1 + (m-n)\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} \\ &\quad + (v - v^{-1}) \sum_{i=2-m}^{m-n} T_{\mathbb{O}_{i, \tau}} + T_{\mathbb{O}_{id, \tau}}. \end{aligned}$$

□

Proposition 2.38. For $i \in \mathbb{N}_+$ and $\tilde{w} \in \mathbb{O}_{i, \tau}$. (1) If $\ell(\mathbb{O}_{i, \tau}) \leq \ell(\tilde{w}) \leq 6i - 5$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E_{(\lfloor \frac{i}{2} \rfloor + 1 + \frac{\ell(\tilde{w}) - \ell(\mathbb{O}_{i, \tau})}{2})\alpha_1 + i\alpha_2, \tau} \\ 1, & \mathbb{O} = \mathbb{O}_{i, \tau} \\ 0, & \text{otherwise} \end{cases}$$

(2) If $\ell(\tilde{w}) = 6i - 3 + 4k$ where $k \in \mathbb{N}$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^{i-1} E_{(2j-1)\alpha_1 + j\alpha_2, \tau} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E_{(2i-1)\alpha_1 + i\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 1 \leq j \leq k - 1, \lambda \in E_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau} \\ (k - 1 - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 2 \leq j \leq k - 2, \lambda \in E'_{(2i+j)\alpha_1 + (i+j)\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau}, \text{ where } 1 \leq j \leq k \\ k(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{l, \tau}, \text{ where } 2 - 2i \leq l \leq i - 1 \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{O}_{i, \tau} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{2-2i-j, \tau} \text{ or } \mathbb{O}_{i+j, \tau}, \text{ where } 1 \leq j \leq k - 1 \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{id, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i-1)\alpha_1 + i\alpha_2, \tau} \\ 0, & \text{otherwise} \end{cases}$$

(3) If $\ell(\tilde{w}) = 6i - 1 + 4k$ where $k \in \mathbb{N}$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (k + 1)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^{i-1} E_{(2j-1)\alpha_1 + j\alpha_2, \tau} \\ (k + 1)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E_{(2i-1)\alpha_1 + i\alpha_2, \tau} \\ (k + 1 - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 1 \leq j \leq k, \lambda \in E_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 2 \leq j \leq k - 1, \lambda \in E'_{(2i+j)\alpha_1 + (i+j)\alpha_2, \tau} \\ (k + 1 - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau}, \text{ where } 1 \leq j \leq k \\ (k + 1)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{l, \tau}, \text{ where } 2 - 2i \leq l \leq i - 1 \\ (k + 1)(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{O}_{i, \tau} \\ (k + 1 - j)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{2-2i-j, \tau} \text{ or } \mathbb{O}_{i+j, \tau}, \text{ where } 1 \leq j \leq k \\ (k + 1)(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{id, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(2i-1)\alpha_1 + i\alpha_2, \tau} \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.39. For $i \in \mathbb{N}$ and $\tilde{w} \in \mathbb{O}_{-i, \tau}$. (1) If $\ell(\mathbb{O}_{-i, \tau}) \leq \ell(\tilde{w}) \leq 6i + 3$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E'_{(i+1)\alpha_1 + (\lfloor \frac{i+1}{2} \rfloor + \frac{\ell(\tilde{w}) - \ell(\mathbb{O}_{-i, \tau})}{2})\alpha_2, \tau} \\ 1, & \mathbb{O} = \mathbb{O}_{-i, \tau} \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\ell(\tilde{w}) = 6i + 5 + 4k$ where $k \in \mathbb{N}$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} k(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^i E'_{j\alpha_1 + (2j-1)\alpha_2, \tau} \\ k(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E'_{(i+1)\alpha_1 + (2i+1)\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 1 \leq j \leq k - 1, \lambda \in E'_{(i+1+j)\alpha_1 + (2i+1+j)\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 2 \leq j \leq k - 1, \lambda \in E_{(i+1+j)\alpha_1 + (2i+2+j)\alpha_2, \tau} \\ (k - j)(v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+1+j)\alpha_1 + (2i+2+j)\alpha_2, \tau}, \text{ where } 1 \leq j \leq k \\ k(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{l, \tau}, \text{ where } 1 - i \leq l \leq 2i + 2 \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{O}_{-i, \tau} \\ (k - j)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{-i-j, \tau} \text{ or } \mathbb{O}_{2i+2+j, \tau}, \text{ where } 1 \leq j \leq k - 1 \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{id, \tau} \\ (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+1)\alpha_1 + (2i+2)\alpha_2, \tau} \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\ell(\tilde{w}) = 6i + 7 + 4k$ where $k \in \mathbb{N}$. Then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (k+1)(v-v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in \sqcup_{j=2}^i E'_{j\alpha_1 + (2j-1)\alpha_2, \tau} \\ (k+1)(v-v^{-1})^3 + (v-v^{-1}), & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } \lambda \in E'_{(i+1)\alpha_1 + (2i+1)\alpha_2, \tau} \\ (k+1-j)(v-v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 1 \leq j \leq k, \lambda \in E'_{(i+1+j)\alpha_1 + (2i+1+j)\alpha_2, \tau} \\ (k+1-j)(v-v^{-1})^3, & \mathbb{O} = \mathbb{O}_{\lambda, \tau}, \text{ where } 2 \leq j \leq k, \lambda \in E_{(i+1+j)\alpha_1 + (2i+2+j)\alpha_2, \tau} \\ (k+1-j)(v-v^{-1})^3 + (v-v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+1+j)\alpha_1 + (2i+2+j)\alpha_2, \tau}, \text{ where } 1 \leq j \leq k \\ (k+1)(v-v^{-1})^2, & \mathbb{O} = \mathbb{O}_{l, \tau}, \text{ where } 1-i \leq l \leq 2i+2 \\ (k+1)(v-v^{-1})^2 + 1, & \mathbb{O} = \mathbb{O}_{-i, \tau} \\ (k+1-j)(v-v^{-1})^2, & \mathbb{O} = \mathbb{O}_{-i-j, \tau} \text{ or } \mathbb{O}_{2i+2+j, \tau}, \text{ where } 1 \leq j \leq k \\ (k+1)(v-v^{-1}), & \mathbb{O} = \mathbb{O}_{id, \tau} \\ (v-v^{-1}), & \mathbb{O} = \mathbb{O}_{(i+1)\alpha_1 + (2i+2)\alpha_2, \tau} \\ 0, & \text{otherwise.} \end{cases}$$

We calculate class polynomials for Proposition 2.38. Symmetrically, we obtain them for Proposition 2.39.

Proof of Proposition 2.38. Since (3) is easily deduced from (2), it is sufficient for us to prove (1) and (2). We set $A = \{t^{k\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau \mid k \geq \lfloor \frac{i}{2} \rfloor + 1\} \sqcup \{t^{(2i-1+k)\alpha_1 + (i+k)\alpha_2} s_2 \tau \mid k \in \mathbb{N}\}$. Then we can check directly that any element in $\mathbb{O}_{i, \tau}$ is \sim to a unique element in A . Hence it is sufficient to consider those elements in A . For (1) It is sufficient to consider that $\tilde{w} = t^{(\lfloor \frac{i}{2} \rfloor + 1 + \frac{\ell(\tilde{w}) - \ell(\mathbb{O}_{i, \tau})}{2})\alpha_1 + i\alpha_2} s_1 s_2 s_1 \tau$, then

$$T_{\tilde{w}} \equiv (v-v^{-1}) \sum_{\substack{E \\ (\lfloor \frac{i}{2} \rfloor + 1 + \frac{\ell(\tilde{w}) - \ell(\mathbb{O}_{i, \tau})}{2})\alpha_1 + i\alpha_2, \tau}} + T_{\mathbb{O}_{id, \tau}}.$$

For (2), it is sufficient to consider that $\tilde{w} = t^{(2i-1+k)\alpha_1 + (i+k)\alpha_2} s_2 \tau$ where $k \in \mathbb{N}$. If $k \leq 3$, we calculate the class polynomials one by one, now if $k \geq 4$, then

$$\begin{aligned} T_{t^{(2i-1+k)\alpha_1 + (i+k)\alpha_2} s_2 \tau} &\equiv (v-v^{-1}) T_{t^{(2i-1+k)\alpha_1 + (i+k)\alpha_2} s_2 s_1 \tau} + (v-v^{-1}) T_{t^{(2i-1+k)\alpha_1 + (i+k-1)\alpha_2} s_1 s_2 \tau} + T_{t^{(2i-2+k)\alpha_1 + (i+k-1)\alpha_2} s_2 \tau} \\ &\dots\dots\dots \\ &\equiv (v-v^{-1}) \sum_{j=3}^k T_{\mathbb{O}_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau}} + (v-v^{-1}) \sum_{j=2}^{k-1} T_{t^{(2i+j)\alpha_1 + (i+j)\alpha_2} s_1 s_2 \tau} + T_{t^{(2i+1)\alpha_1 + (i+2)\alpha_2} s_2 \tau} \end{aligned}$$

We go further to obtain the class polynomials from the righthand side which is actually

$$\begin{aligned} &k(v-v^{-1})^3 \sum_{\lambda \in \sqcup_{j=2}^{i-1} E_{(2j-1)\alpha_1 + j\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + (k(v-v^{-1})^3 + (v-v^{-1})) \sum_{\lambda \in E_{(2i-1)\alpha_1 + i\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} \\ &+ \sum_{j=1}^{k-1} (k-j)(v-v^{-1})^3 \sum_{\lambda \in E_{(2j-1+j)\alpha_1 + (i+j)\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} + \sum_{j=2}^{k-2} (k-1-j)(v-v^{-1})^3 \sum_{\lambda \in E'_{(2i+j)\alpha_1 + (i+j)\alpha_2, \tau}} T_{\mathbb{O}_{\lambda, \tau}} \\ &+ \sum_{j=1}^k ((k-j)(v-v^{-1})^3 + (v-v^{-1})) T_{\mathbb{O}_{(2i-1+j)\alpha_1 + (i+j)\alpha_2, \tau}} + \sum_{j=2-2i}^{i-1} k(v-v^{-1})^2 T_{\mathbb{O}_{j, \tau}} + (k(v-v^{-1})^2 + 1) T_{\mathbb{O}_{i, \tau}} \end{aligned}$$

$$+(v-v^{-1})T_{\mathbb{O}_{(2i-1)\alpha_1+i\alpha_2,\tau}}+\sum_{j=1}^{k-1}(k-j)(v-v^{-1})^2(T_{\mathbb{O}_{2-2i-j,\tau}}+T_{\mathbb{O}_{i+j,\tau}})+k(v-v^{-1})T_{\mathbb{O}_{id,\tau}}.$$

□

2.3. The quasi-split case. At first we classify the δ -conjugacy classes in \widetilde{W} . In this section we set \cdot to be the usual \widetilde{W} -conjugation (i.e. $x \cdot y = xyx^{-1}$) and \cdot_δ be the δ -conjugation (i.e., $x \cdot y = xy\delta(x)^{-1}$). Let

$$\begin{aligned} \mathbb{O}_{0,\delta} = & \{t^{k(\alpha_1+2\alpha_2)}s_2s_1, t^{k(2\alpha_1+\alpha_2)}s_1s_2, t^{k(\alpha_1-\alpha_2)} \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau \cdot t^{k(\alpha_1+2\alpha_2)}s_2s_1)\tau^2, (\tau \cdot t^{k(2\alpha_1+\alpha_2)}s_1s_2)\tau^2, (\tau \cdot t^{k(\alpha_1-\alpha_2)})\tau^2 \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{k(\alpha_1+2\alpha_2)}s_2s_1)\tau, (\tau^2 \cdot t^{k(2\alpha_1+\alpha_2)}s_1s_2)\tau, (\tau^2 \cdot t^{k(\alpha_1-\alpha_2)})\tau \mid k \in \mathbb{Z}\}; \end{aligned}$$

$$\begin{aligned} \mathbb{O}_{1,\delta} = & \{t^\lambda s_1, t^\lambda s_2 \mid \lambda \in \mathcal{Q}\} \sqcup \{(\tau \cdot t^\lambda s_1)\tau^2, (\tau \cdot t^\lambda s_2)\tau^2 \mid \lambda \in \mathcal{Q}\} \\ & \sqcup \{(\tau^2 \cdot t^\lambda s_1)\tau, (\tau^2 \cdot t^\lambda s_2)\tau \mid \lambda \in \mathcal{Q}\}; \end{aligned}$$

$$\begin{aligned} \mathbb{O}'_{1,\delta} = & \{t^{k\alpha_1+(2i+1)\alpha_2}s_1s_2s_1, t^{(2k+1)\alpha_1+2i\alpha_2}s_1s_2s_1 \mid k, i \in \mathbb{Z}\} \\ & \sqcup \{(\tau \cdot t^{k\alpha_1+(2i+1)\alpha_2}s_1s_2s_1)\tau^2, (\tau \cdot t^{(2k+1)\alpha_1+2i\alpha_2}s_1s_2s_1)\tau^2 \mid k, i \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{k\alpha_1+(2i+1)\alpha_2}s_1s_2s_1)\tau, (\tau^2 \cdot t^{(2k+1)\alpha_1+2i\alpha_2}s_1s_2s_1)\tau \mid k, i \in \mathbb{Z}\}; \end{aligned}$$

$$\begin{aligned} \mathbb{O}_{3,\delta} = & \{t^{2k\alpha_1+2i\alpha_2}s_1s_2s_1 \mid k, i \in \mathbb{Z}\} \sqcup \{(\tau \cdot t^{2k\alpha_1+2i\alpha_2}s_1s_2s_1)\tau^2 \mid k, i \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{2k\alpha_1+2i\alpha_2}s_1s_2s_1)\tau \mid k, i \in \mathbb{Z}\}; \end{aligned}$$

For $m \in \mathbb{N}_+$, we set

$$\epsilon(m) = \begin{cases} 1, & m \text{ odd} \\ 0, & m \text{ even.} \end{cases}$$

$$\begin{aligned} \mathbb{O}_{2m,\delta} = & \{t^{(k-\lfloor \frac{m}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1, t^{(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1 \mid k \in \mathbb{Z}\} \\ & \sqcup \{t^{(2k+\epsilon(m))\alpha_1+(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_2}s_1s_2, t^{(2k+\epsilon(m))\alpha_1+(k-\lfloor \frac{m}{2} \rfloor)\alpha_2}s_1s_2 \mid k \in \mathbb{Z}\} \\ & \sqcup \{t^{k\alpha_1+(m-k)\alpha_2}, t^{(k-m)\alpha_1-k\alpha_2} \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau \cdot t^{(k-\lfloor \frac{m}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1)\tau^2, (\tau \cdot t^{(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1)\tau^2 \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau \cdot t^{(2k+\epsilon(m))\alpha_1+(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_2}s_1s_2)\tau^2, (\tau \cdot t^{(2k+\epsilon(m))\alpha_1+(k-\lfloor \frac{m}{2} \rfloor)\alpha_2}s_1s_2)\tau^2 \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau \cdot t^{k\alpha_1+(m-k)\alpha_2})\tau^2, (\tau \cdot t^{(k-m)\alpha_1-k\alpha_2})\tau^2 \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{(k-\lfloor \frac{m}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1)\tau, (\tau^2 \cdot t^{(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_1+(2k+\epsilon(m))\alpha_2}s_2s_1)\tau \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{(2k+\epsilon(m))\alpha_1+(k+\lfloor \frac{m+1}{2} \rfloor)\alpha_2}s_1s_2)\tau, (\tau^2 \cdot t^{(2k+\epsilon(m))\alpha_1+(k-\lfloor \frac{m}{2} \rfloor)\alpha_2}s_1s_2)\tau \mid k \in \mathbb{Z}\} \\ & \sqcup \{(\tau^2 \cdot t^{k\alpha_1+(m-k)\alpha_2})\tau, (\tau^2 \cdot t^{(k-m)\alpha_1-k\alpha_2})\tau \mid k \in \mathbb{Z}\}. \end{aligned}$$

Remark 2.40. Let $\tilde{w} \in \widetilde{W}$ then $\tilde{w} = w_a\tau'$ where $w_a \in W_a$ and $\tau' \in \Omega$. Thus $\tilde{w} \sim w_a$. In the following, to consider $\tilde{w} \in \widetilde{W}$, it is sufficient to consider $w_a \in W_a$.

Theorem 2.41. $\mathbb{O}_{0,\delta}$, $\mathbb{O}_{1,\delta}$, $\mathbb{O}'_{1,\delta}$, $\mathbb{O}_{3,\delta}$, $\mathbb{O}_{2m,\delta}$ where $m \in \mathbb{N}_+$ form all the δ -conjugacy classes of \widetilde{W} . Moreover,

$$\begin{aligned}\ell(\mathbb{O}_{0,\delta}) &= 0, \quad \ell(\mathbb{O}_{3,\delta}) = 3, \\ \ell(\mathbb{O}_{1,\delta}) &= \ell(\mathbb{O}'_{1,\delta}) = 1, \\ \ell(\mathbb{O}_{2m,\delta}) &= 2m.\end{aligned}$$

Proof. By the definitions of $\mathbb{O}_{0,\delta}$, $\mathbb{O}_{1,\delta}$, $\mathbb{O}'_{1,\delta}$, $\mathbb{O}_{3,\delta}$, $\mathbb{O}_{2m,\delta}$ for $m \in \mathbb{N}_+$, we have $\widetilde{W} = \mathbb{O}_{0,\delta} \sqcup \mathbb{O}_{1,\delta} \sqcup \mathbb{O}'_{1,\delta} \sqcup \mathbb{O}_{3,\delta} \sqcup \bigsqcup_{m \in \mathbb{N}_+} \mathbb{O}_{2m,\delta}$. And the theorem is proved together with Lemma 2.42. \square

Lemma 2.42. (1) $\mathbb{O}_{0,\delta}$ is the δ -conjugacy class of \widetilde{W} with minimal length 0;
(2) $\mathbb{O}_{1,\delta}$ and $\mathbb{O}'_{1,\delta}$ are the δ -conjugacy classes of \widetilde{W} with minimal length 1;
(3) $\mathbb{O}_{3,\delta}$ is the δ -conjugacy class of \widetilde{W} with minimal length 3;
(4) If $m \in \mathbb{N}_+$, then $\mathbb{O}_{2m,\delta}$ is the δ -conjugacy class of \widetilde{W} with minimal length $2m$.

Proof. We prove (1) and others are similar. We check easily that for any $\tilde{w} \in \widetilde{W}$, then $\tilde{w} \cdot_\delta \mathbb{O}_{0,\delta} \subset \mathbb{O}_{0,\delta}$. Then we need to show that for any element $\tilde{w} \in \mathbb{O}_{0,\delta}$, \tilde{w} is δ -conjugate to id . By 2.40, it is sufficient to consider that $\tilde{w} \in \{t^{k(\alpha_1+2\alpha_2)}s_2s_1, t^{k(2\alpha_1+\alpha_2)}s_1s_2, t^{k(\alpha_1-\alpha_2)} \mid k \in \mathbb{Z}\}$ and we use induction on the length. If $\tilde{w} = id$, there is nothing to prove. If $\tilde{w}' \in \{t^{k(\alpha_1+2\alpha_2)}s_2s_1, t^{k(2\alpha_1+\alpha_2)}s_1s_2, t^{k(\alpha_1-\alpha_2)} \mid k \in \mathbb{Z}\}$, and $\ell(\tilde{w}') < \ell(\tilde{w})$, then \tilde{w}' is δ -conjugate to id . If $\tilde{w} = t^{k(\alpha_1+2\alpha_2)}s_2s_1$ or $t^{k(2\alpha_1+\alpha_2)}s_1s_2$ where $k \in \mathbb{N}_+$, and we set $\tilde{w}_1 = s_0\tilde{w}s_0$, then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus \tilde{w}_1 is δ -conjugate to id , so is \tilde{w} . If $\tilde{w} = t^{k(\alpha_1-\alpha_2)}$ or $t^{-k(\alpha_1+2\alpha_2)}s_2s_1$ where $k \in \mathbb{N}_+$, and we set $\tilde{w}_1 = s_2\tilde{w}s_1$ then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus \tilde{w}_1 is δ -conjugate to id , so is \tilde{w} . If $\tilde{w} = t^{k(\alpha_2-\alpha_1)}$ or $t^{-k(2\alpha_1+\alpha_2)}s_1s_2$ where $k \in \mathbb{N}_+$, and we set $\tilde{w}_1 = s_1\tilde{w}s_2$ then $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, thus \tilde{w}_1 is δ -conjugate to id , so is \tilde{w} . Hence (1) is proved. \square

Theorem 2.43. Let $\tilde{w}, \tilde{w}' \in \mathbb{O}_{i,\delta}$ where $i = 0, 1$ or $2m$ where $m \in \mathbb{N}_+$. If $\ell(\tilde{w}) = \ell(\tilde{w}')$, then $T_{\tilde{w}} \equiv T_{\tilde{w}'}$.

Proof. We use induction on the length ℓ . We first prove that $\tilde{w}, \tilde{w}' \in \mathbb{O}_{1,\delta}$. If $\ell(\tilde{w}) = \ell(\tilde{w}') = 1$, obviously. Now if $\ell(\tilde{w}) = \ell(\tilde{w}') = 2k + 1$ where $k \in \mathbb{N}_+$ and we assume for any $\tilde{w}_1, \tilde{w}'_1 \in \mathbb{O}_{1,\delta}$ and $\ell(\tilde{w}_1) = \ell(\tilde{w}'_1) < 2k + 1$ then $T_{\tilde{w}_1} \equiv T_{\tilde{w}'_1}$. It is sufficient to show that if $\tilde{w}_1 = s_i\tilde{w}\delta(s_i)$ and $\ell(\tilde{w}_1) = \ell(\tilde{w}) - 2$, then $s_i\tilde{w}$ is a minimal length element in $\mathbb{O}_{2k,\delta}$. If $k = 3j + 3$, then $\tilde{w} = t^{(j+2)\alpha_1-j\alpha_2+i(\alpha_1+2\alpha_2)}s_1$ where $0 \leq i \leq j$. If $k = 3j + 4$, then $\tilde{w} = t^{(j+2)\alpha_1-(j+1)\alpha_2+i(\alpha_1+2\alpha_2)}s_1$ where $0 \leq i \leq j + 1$. If $k = 3j + 5$, then $\tilde{w} = t^{(j+4)\alpha_1-(j+1)\alpha_2+i(\alpha_1+2\alpha_2)}s_1$ where $0 \leq i \leq j + 1$. Thus $s_0\tilde{w} \sim \tilde{w}s_0$ and $\tilde{w}s_0$ is a minimal length element of $\mathbb{O}_{2k,\delta}$. Similar argument for other cases. For \tilde{w} and \tilde{w}' are contained in $\mathbb{O}_{0,\delta}$. For $k \in \mathbb{N}_+$, and if $\ell(\tilde{w}) = 6k - 4$, then $\tilde{w} = t^{(1-k)(\alpha_1+2\alpha_2)}s_2s_1$ or $\tilde{w} = t^{(1-k)(2\alpha_1+\alpha_2)}s_1s_2$. If $\ell(\tilde{w}) = 6k - 2$, then $\tilde{w} = t^{k(\alpha_1+2\alpha_2)}s_2s_1$ or $\tilde{w} = t^{k(2\alpha_1+\alpha_2)}s_1s_2$. If $\ell(\tilde{w}) = 6k$, then $\tilde{w} = t^{k(\alpha_1-\alpha_2)}$ or $\tilde{w} = t^{k(\alpha_2-\alpha_1)}$. By symmetry, we know $T_{t^{(1-k)(\alpha_1+2\alpha_2)}s_2s_1} \equiv T_{t^{(1-k)(2\alpha_1+\alpha_2)}s_1s_2}$, $T_{t^{k(\alpha_1+2\alpha_2)}s_2s_1} \equiv T_{t^{k(2\alpha_1+\alpha_2)}s_1s_2}$ or $T_{t^{k(\alpha_1-\alpha_2)}} \equiv T_{t^{k(\alpha_2-\alpha_1)}}$. Thus for $\mathbb{O}_{0,\delta}$ is proved. If

$\tilde{w} \in \mathbb{O}_{2m,\delta}$ and $\ell(\tilde{w}) = 2m + 2k$ for some $k \in \mathbb{N}_+$. If $\tilde{w} = t^\lambda s_2 s_1$, then we have $\tilde{w} \xrightarrow{s_0}_\delta s_0 \tilde{w} s_0$ or $\tilde{w} \xrightarrow{s_2}_\delta s_2 \tilde{w} s_1$. In either case, $T_{\tilde{w}} \equiv (v - v^{-1})T_{s_i \tilde{w}} + T_{s_i \tilde{w} \delta(s_i)}$ where $i = 0$ or 2 , $s_i \tilde{w} \in \mathbb{O}_{1,\delta}$. Similar argument for \tilde{w}' where $\ell(\tilde{w}) = \ell(\tilde{w}')$. We have some i such that $T_{\tilde{w}'} \equiv (v - v^{-1})T_{s_i' \tilde{w}'} + T_{s_i' \tilde{w}' \delta(s_i')}$ and $s_i' \tilde{w}' \in \mathbb{O}_{1,\delta}$. Then by the proof for $\mathbb{O}_{1,\delta}$ and induction we have $T_{\tilde{w}} \equiv T_{\tilde{w}'}$. \square

Proposition 2.44. *If $\tilde{w} \in \mathbb{O}_{1,\delta}$ and $\ell(\tilde{w}) = 2k + 1$ where $k \in \mathbb{N}$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2j,\delta} \ (1 \leq j \leq k) \\ 1, & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows directly from the proof of Lemma 2.42 and Theorem 2.43. \square

Proposition 2.45. *If $\tilde{w} \in \mathbb{O}_{0,\delta}$ and $\ell(\tilde{w}) = 2k$ where $k \in \mathbb{N}$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (k - j)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{2j,\delta} \ (1 \leq j \leq k - 1) \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 1, & \mathbb{O} = \mathbb{O}_{0,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For $\tilde{w} \in \mathbb{O}_{0,\delta}$ and $\ell(\tilde{w}) = 2k$ where $k \in \mathbb{N}$. If $k = 0$, it is obvious. For $k > 0$, there exist s_i with $i = 0, 1$ or 2 such that

$$T_{\tilde{w}} \equiv (v - v^{-1})T_{s_i \tilde{w}} + T_{s_i \tilde{w} \delta(s_i)}.$$

Here $s_i \tilde{w} \in \mathbb{O}_{1,\delta}$ with $\ell(s_i \tilde{w}) = 2k - 1$ and $\ell(s_i \tilde{w} \delta(s_i)) = 2k - 2$. Use Proposition 2.44 and calculate inductively we have

$$T_{\tilde{w}} \equiv \sum_{j=1}^{k-1} (k - j)(v - v^{-1})^2 T_{\mathbb{O}_{2j,\delta}} + k(v - v^{-1})T_{\mathbb{O}_{1,\delta}} + T_{\mathbb{O}_{0,\delta}}.$$

\square

Proposition 2.46. *Let $m \in \mathbb{N}_+$, if $\tilde{w} \in \mathbb{O}_{2m,\delta}$ and $\ell(\tilde{w}) = 2m + 2k$ where $k \in \mathbb{N}$. Then*

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (k - j)(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{2m+2j,\delta} \ (1 \leq j \leq k - 1) \\ k(v - v^{-1})^2 + 1, & \mathbb{O} = \mathbb{O}_{2m,\delta} \\ k(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{2m-2j,\delta} \ (1 \leq j \leq m - 1) \\ k(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is quite similar to that of Proposition 2.45 and we will omit it. \square

If $\tilde{w}, \tilde{w}' \in \mathbb{O}'_{1,\delta}$ lie in the critical strip and $\ell(\tilde{w}) = \ell(\tilde{w}')$, then $T_{\tilde{w}} \equiv T_{\tilde{w}'}$. Moreover, if $\tilde{w} \in \mathbb{O}'_{1,\delta}$ with $\ell(\tilde{w}) = 2k + 1$ where $k \in \mathbb{N}$, then

$$f_{\tilde{w},\mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2j,\delta}, \ 1 \leq j \leq k \\ 1, & \mathbb{O} = \mathbb{O}'_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

If $\tilde{w} \in \mathbb{O}'_{1,\delta}$ lies in the shrunkun Weyl chambers, we do not have a uniform formula for those class polynomials. For example, if $\tilde{w} = t^{2\alpha_1 + \alpha_2} s_1 s_2 s_1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^3 + 2(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2,\delta} \\ (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 1, & \mathbb{O} = \mathbb{O}'_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

If $\ell(\tilde{w}) = 7$ and $\tilde{w} = t^{\alpha_1 - \alpha_2} s_1 s_2 s_1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{6,\delta} \\ (v - v^{-1})^3 + 2(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2,\delta} \\ (v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 1, & \mathbb{O} = \mathbb{O}'_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

If $\ell(\tilde{w}) = 7$ and $\tilde{w} = t^{-\alpha_1 - \alpha_2} s_1 s_2 s_1$, then

$$f_{\tilde{w}, \mathbb{O}} = \begin{cases} (v - v^{-1})^3 + (v - v^{-1}), & \mathbb{O} = \mathbb{O}_{4,\delta} \\ (v - v^{-1})^3 + 2(v - v^{-1}), & \mathbb{O} = \mathbb{O}_{2,\delta} \\ 2(v - v^{-1})^2, & \mathbb{O} = \mathbb{O}_{1,\delta} \\ 1, & \mathbb{O} = \mathbb{O}'_{1,\delta} \\ 0, & \text{otherwise.} \end{cases}$$

Inductively, if $f_{\tilde{w}, \mathbb{O}} \neq 0$, we deduce that $\deg f_{\tilde{w}, \mathbb{O}_{1,\delta}} = 2$ and $\deg f_{\tilde{w}, \mathbb{O}_{2m,\delta}} = 3$ or 1 for certain m . Moreover, $f_{\tilde{w}, \mathbb{O}'_{1,\delta}} = 1$.

Similar argument for $\tilde{w} \in \mathbb{O}_{3,\delta}$.

3. APPLICATIONS

3.1. Affine Deligne-Lusztig varieties of basic elements. First we assume that $b \in PGL_3(L)$ and $\tilde{w} \in \tilde{W}$.

Theorem 3.1. (1) *If $b = 1$, then the affine Deligne-Lusztig variety $X_{\tilde{w}}(b) \neq \emptyset$ if and only if \tilde{w} satisfies one of the following conditions:*

- (a) $\tilde{w} = id$
- (b) $\tilde{w} \in \mathbb{O}_1$ or $\tilde{w} \in \mathbb{O}_2$
- (c) $\tilde{w} \in \mathbb{C}_i$ or \mathbb{C}'_i , where $i \in \mathbb{N}_+$ and $\ell(\tilde{w}) \geq 6i + 3$.

(2) *If $b = \tau$, then the affine Deligne-Lusztig variety $X_{\tilde{w}}(b) \neq \emptyset$ if and only if \tilde{w} satisfies one of the following conditions:*

- (a) $\tilde{w} \in \mathbb{O}_{id,\tau}$
- (b) $\tilde{w} \in \mathbb{O}_{i,\tau}$, where $i \in \mathbb{N}_+$ and $\ell(\tilde{w}) \geq 6i - 1$
- (c) $\tilde{w} \in \mathbb{O}_{1-i,\tau}$, where $\ell(\tilde{w}) \geq 6i + 1$.

Proof. Following the “Dimension=Degree” Theorem 1.11, we go back to §2 to check those nonzero class polynomials. \square

Once the affine Deligne-Lusztig variety $X_{\tilde{w}}(b) \neq \emptyset$, we do have a neat dimension formula.

Theorem 3.2. (1) If $b = 1$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} 0, & \tilde{w} = Id \\ 1, & \tilde{w} \in \mathbb{O}_1 \text{ and } \ell(\tilde{w}) = 1 \\ \frac{\ell(\tilde{w})}{2} + 1, & \tilde{w} \in \mathbb{O}_2 \\ \frac{\ell(\tilde{w})+3}{2}, & \tilde{w} \in \mathbb{O}_1 \text{ with } \ell(\tilde{w}) > 1, \text{ or} \\ & \tilde{w} \in \mathbb{C}_i \text{ or } \mathbb{C}'_i \text{ for } i \in \mathbb{N}_+. \end{cases}$$

(2) If $b = \tau$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{\ell(\tilde{w})}{2}, & \tilde{w} \in \mathbb{O}_{id,\tau} \\ \frac{\ell(\tilde{w})+1}{2}, & \tilde{w} \in \mathbb{O}_{i,\tau} \text{ where } i \in \mathbb{Z}. \end{cases}$$

Proof. By the “Dimension=Degree” theorem, if $b = 1$, then

$$\dim X_{\tilde{w}}(1) = \frac{1}{2} \max\{\ell(\tilde{w}) + \deg f_{\tilde{w},Id}, \ell(\tilde{w}) + 1 + \deg f_{\tilde{w},\mathbb{O}_1}, \ell(\tilde{w}) + 2 + \deg f_{\tilde{w},\mathbb{O}_2}\}.$$

We check the degree of those class polynomials in §2.1, and the theorem is proved. \square

Remark 3.3. Symmetrically, we have a similar description of the emptiness/nonemptiness pattern and dimension formula of $X_{\tilde{w}}(b)$ for \tilde{w} , $b = \tau^2$.

In the following, the proofs of theorems of the emptiness/nonemptiness pattern and dimension formula are similar to that of Theorem 3.1 and 3.2, and we will omit them. Now we assume that $b \in U_3(L)$.

Theorem 3.4. If b is basic, then $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $\tilde{w} \notin \bigsqcup_{m \in \mathbb{N}_+} \mathbb{O}_{2m,\delta}^{\min}$, where $\mathbb{O}_{2m,\delta}^{\min}$ is the set of minimal length elements of $\mathbb{O}_{2m,\delta}$.

Theorem 3.5. If $b = 1$ and if $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} 0, & \tilde{w} \in \mathbb{O}_{0,\delta} \text{ and } \ell(\tilde{w}) = 0 \\ \frac{1}{2}(\ell(\tilde{w}) + 1), & \tilde{w} \in \mathbb{O}'_{1,\delta} \text{ and lies in critical strips, or } \tilde{w} \in \mathbb{O}_{1,\delta} \\ \frac{1}{2}(\ell(\tilde{w}) + 2), & \tilde{w} \in \mathbb{O}_{0,\delta} \text{ with } \ell(\tilde{w}) > 0, \text{ or } \tilde{w} \in \mathbb{O}_{2m,\delta} \text{ where } m \in \mathbb{N}_+ \\ \frac{1}{2}(\ell(\tilde{w}) + 3), & \tilde{w} \in \mathbb{O}'_{1,\delta} \text{ corresponds to shrunkun Weyl chambers, or} \\ & \tilde{w} \in \mathbb{O}_{3,\delta}. \end{cases}$$

3.2. Affine Deligne-Lusztig varieties of nonbasic elements. By the “Dimension=Degree” theorem, for any $b \in G(L)$ and $\tilde{w} \in \tilde{W}$, the affine Deligne-Lusztig variety $X_{\tilde{w}}(b) \neq \emptyset$ if and only if the corresponding class polynomial is nonzero. Thus for nonbasic b , we check the corresponding class polynomial in Section §2 and we know the emptiness/nonemptiness pattern. For example, if $b \in PGL_3(L)$ corresponds to \mathbb{O}_{λ_0} (i.e. $f(b) = f(\mathbb{O}_{\lambda_0})$) where $\lambda_0 \in P_+ \cap Q_{sh}$, then $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $f_{\tilde{w},\mathbb{O}_{\lambda_0}} \neq 0$. If $b \in U_3(L)$ corresponds to $\mathbb{O}_{2m_0,\delta}$ where $m_0 \in \mathbb{N}_+$, then $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $f_{\tilde{w},\mathbb{O}_{2m_0,\delta}} \neq 0$.

Theorem 3.6. (1) If $b \in PGL_3(L)$ corresponds to \mathbb{O}_{λ_0} where $\lambda_0 \in P_+ \cap Q_{sh}$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0})) - \langle \lambda_0, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_{\lambda_0} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0}) + 1) - \langle \lambda_0, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_1 \text{ and } \lambda_0 = \frac{\ell(\tilde{w})-1}{4}(\alpha_1 + \alpha_2) \text{ or} \\ & \tilde{w} \in \mathbb{C}_i \text{ or } \mathbb{C}'_i \text{ } i \in \mathbb{N}_+ \text{ with } \ell(\tilde{w}) \leq 6i + 1 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0}) + 2) - \langle \lambda_0, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_2 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0}) + 3) - \langle \lambda_0, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_1 \text{ and } \lambda_0 \neq \frac{\ell(\tilde{w})-1}{4}(\alpha_1 + \alpha_2) \text{ or} \\ & \tilde{w} \in \mathbb{C}_i \text{ or } \mathbb{C}'_i \text{ } i \in \mathbb{N}_+ \text{ with } \ell(\tilde{w}) > 6i + 1 \end{cases}$$

(2) If $b \in PGL_3(L)$ corresponds to $\mathbb{O}_{i_0(\alpha_1+2\alpha_2)}$ where $i_0 \in \mathbb{N}_+$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + 6i_0 + 1) - t_0, & \tilde{w} \in \mathbb{C}_i \text{ or } \mathbb{C}'_i \text{ with } \ell(\tilde{w}) \leq 6i + 1, i \in \mathbb{N}_+ \\ \frac{1}{2}(\ell(\tilde{w}) + 6i_0 + 2) - t_0, & \tilde{w} \in \mathbb{O}_2 \\ \frac{1}{2}(\ell(\tilde{w}) + 6i_0 + 3) - t_0, & \tilde{w} \in \mathbb{O}_1 \text{ or } \tilde{w} \in \mathbb{C}_i, \mathbb{C}'_i \text{ with } \ell(\tilde{w}) > 6i + 1. \end{cases}$$

Where $t_0 = i_0\langle \alpha_1 + 2\alpha_2, 2\rho \rangle$.

Symmetrically, if b corresponds to $\mathbb{O}_{i_0(2\alpha_1+\alpha_2)}$, then there is a similar dimension formula.

(3) If $b \in PGL_3(L)$ corresponds to \mathbb{C}_{i_0} where $i_0 \in \mathbb{N}_+$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{C}_{i_0})) - t'_0, & \tilde{w} \in \mathbb{C}_i \text{ or } \mathbb{C}'_i \text{ with } \ell(\tilde{w}) \leq 6i + 1, i \in \mathbb{N}_+ \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{C}_{i_0}) + 1) - t'_0, & \tilde{w} \in \mathbb{O}_2 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{C}_{i_0}) + 2) - t'_0, & \tilde{w} \in \mathbb{O}_1 \text{ or } \tilde{w} \in \mathbb{C}_i, \mathbb{C}'_i \text{ with } \ell(\tilde{w}) > 6i + 1, i \in \mathbb{N}_+. \end{cases}$$

Where $t'_0 = i_0\langle \alpha_1 + 2\alpha_2, \rho \rangle$.

Symmetrically, when b corresponds to \mathbb{C}'_{i_0} we have a similar dimension formula.

Theorem 3.7. (1) Let $b \in PGL_3(L)$. If b corresponds to $\mathbb{O}_{\lambda_0, \tau}$ where $\lambda_0 \in P_+ \cap Q_{sh}$ but $\lambda_0 \neq (2i-1)\alpha_1 + i\alpha_2$ for $i \in \mathbb{N}_+$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(\tau) = \begin{cases} \ell(\tilde{w}) - \langle \bar{v}_{\mathbb{O}_{\lambda_0, \tau}}, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_{\lambda_0, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0, \tau}) + 1) - \langle \bar{v}_{\mathbb{O}_{\lambda_0, \tau}}, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_{i, \tau} \text{ or } \mathbb{O}_{1-i, \tau} \text{ and } \ell(\tilde{w}) \leq 6i - 3 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0, \tau}) + 2) - \langle \bar{v}_{\mathbb{O}_{\lambda_0, \tau}}, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{\lambda_0, \tau}) + 3) - \langle \bar{v}_{\mathbb{O}_{\lambda_0, \tau}}, 2\rho \rangle, & \tilde{w} \in \mathbb{O}_{i, \tau} \text{ or } \mathbb{O}_{1-i, \tau} \text{ and } \ell(\tilde{w}) > 6i - 3. \end{cases}$$

(2) Let $b \in PGL_3(L)$. If b corresponds to $\mathbb{O}_{i_0(\alpha_1+2\alpha_2), \tau}$ where $i_0 \in \mathbb{N}_+$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2i_0, \tau})) - l_0, & \tilde{w} \in \mathbb{O}_{i, \tau}, i \in \mathbb{N}_+ \text{ and } \ell(\tilde{w}) \leq 6i - 3 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2i_0, \tau}) + 1) - l_0, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2i_0, \tau}) + 2) - l_0, & \tilde{w} \in \mathbb{O}_{i, \tau} \text{ and } \ell(\tilde{w}) > 6i - 3 \text{ or } \tilde{w} \in \mathbb{O}_{1-i, \tau}. \end{cases}$$

Where $l_0 = \langle (i_0 - \frac{1}{3})(\alpha_1 + 2\alpha_2), 2\rho \rangle$.

(3) Let $b \in PGL_3(L)$. If b corresponds to $\mathbb{O}_{(2i_0-1)\alpha_1+i_0\alpha_2, \tau}$ where $i_0 \in \mathbb{N}_+$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2(1-i_0), \tau})) - l_1, & \tilde{w} \in \mathbb{O}_{1-i, \tau}, i \in \mathbb{N}_+ \text{ and } \ell(\tilde{w}) \leq 6i - 1 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2(1-i_0), \tau}) + 1) - l_1, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{2(1-i_0), \tau}) + 2) - l_1, & \tilde{w} \in \mathbb{O}_{1-i, \tau} \text{ and } \ell(\tilde{w}) > 6i - 1 \text{ or } \tilde{w} \in \mathbb{O}_{i, \tau}. \end{cases}$$

Where $l_1 = \langle (i_0 - \frac{2}{3})(2\alpha_1 + \alpha_2), 2\rho \rangle$.

(4) Let $b \in PGL_3(L)$. If b corresponds to $\mathbb{O}_{i_0, \tau}$ where $i_0 \in \mathbb{N}_+$. If $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{i_0, \tau})) - l_2, & \tilde{w} \in \mathbb{O}_{i, \tau} i \in \mathbb{N}_+ \text{ and } \ell(\tilde{w}) \leq 6i - 3 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{i_0, \tau}) + 1) - l_2, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{i_0, \tau}) + 2) - l_2, & \tilde{w} \in \mathbb{O}_{i, \tau} \text{ and } \ell(\tilde{w}) > 6i - 3 \text{ or } \tilde{w} \in \mathbb{O}_{1-i, \tau}. \end{cases}$$

Where $l_2 = \langle (\frac{i_0}{2} - \frac{1}{3})(\alpha_1 + 2\alpha_2), 2\rho \rangle$.

Similarly, if b corresponds to $\mathbb{O}_{1-i_0, \tau}$ and $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{1-i_0, \tau})) - l_3, & \tilde{w} \in \mathbb{O}_{1-i, \tau} i \in \mathbb{N}_+ \text{ and } \ell(\tilde{w}) \leq 6i - 1 \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{1-i_0, \tau}) + 1) - l_3, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ \frac{1}{2}(\ell(\tilde{w}) + \ell(\mathbb{O}_{1-i_0, \tau}) + 2) - l_3, & \tilde{w} \in \mathbb{O}_{1-i, \tau} \text{ and } \ell(\tilde{w}) > 6i - 1 \text{ or } \tilde{w} \in \mathbb{O}_{i, \tau}. \end{cases}$$

Where $l_3 = \langle (\frac{i_0}{2} - \frac{1}{6})(2\alpha_1 + \alpha_2), 2\rho \rangle$.

Theorem 3.8. Let $b \in U_3(L)$. If b corresponds to $\mathbb{O}_{2m_0, \delta}$ where $m_0 \in \mathbb{N}_+$, then $\bar{v}_{\mathbb{O}_{2m_0, \delta}} = \frac{m_0}{2}(\alpha_1 + \alpha_2)$ and $\langle \bar{v}_{\mathbb{O}_{2m_0, \delta}}, 2\rho \rangle = 2m_0$. If $X_{\tilde{w}}(b) \neq \emptyset$, then

$$\dim X_{\tilde{w}}(b) = \begin{cases} 0, & \tilde{w} \in \mathbb{O}_{2m_0, \delta} \text{ and } \ell(\tilde{w}) = 2m_0 \\ \frac{1}{2}(\ell(\tilde{w}) + 1) - m_0, & \tilde{w} \in \mathbb{O}_{1, \delta}, \text{ or} \\ & \tilde{w} \in \mathbb{O}'_{1, \delta} \text{ and lies in critical strips, or} \\ & \tilde{w} \in \mathbb{O}'_{1, \delta} \text{ or } \mathbb{O}_{3, \delta} \text{ with } \ell(\tilde{w}) = 2m_0 + 1 \\ \frac{1}{2}(\ell(\tilde{w}) + 2) - m_0, & \tilde{w} \in \mathbb{O}_{0, \delta} \text{ or } \tilde{w} \in \mathbb{O}_{2m_0, \delta} \text{ and } \ell(\tilde{w}) > 2m_0, \text{ or} \\ & \tilde{w} \in \mathbb{O}_{2m, \delta} \text{ where } m \neq m_0 \\ \frac{1}{2}(\ell(\tilde{w}) + 3) - m_0, & \ell(\tilde{w}) > 2m_0 + 1, \tilde{w} \in \mathbb{O}_{3, \delta} \text{ or} \\ & \tilde{w} \in \mathbb{O}'_{1, \delta} \text{ and corresponds to shrunkun Weyl chambers.} \end{cases}$$

3.3. Affine Deligne-Lusztig varieties: GL_3 and \mathbb{D}_3^\times cases. In general, there is a canonical projection $\pi : GL_n(L) \rightarrow PGL_n(L)$. Let I_1 be the Iwahori subgroup of $GL_n(L)$ described in Chapter 2, and let I_2 be the corresponding Iwahori subgroup of $PGL_n(L)$. Let \tilde{W}_1 (or \tilde{W}_2 , respectively) be the Iwahori-Weyl group. Let $\tilde{w} \in \tilde{W}_1$ and $b \in GL_n(L)$, such that $\kappa(\tilde{w}) = \kappa(b)$. Then the affine Deligne-Lusztig variety $X_{\tilde{w}}(b)$ in the affine flag variety of $GL_n(L)$ is isomorphic to the affine Deligne-Lusztig variety $X_{\pi(\tilde{w})}(\pi(b))$ in the affine flag variety of $PGL_n(L)$ (see [GHN] for more details). Further more, we have $f_{\tilde{w}, \mathbb{O}} = f_{\pi(\tilde{w}), \pi(\mathbb{O})}$. Since we already know the emptiness/nonemptiness pattern and the dimension formulas of the affine Deligne-Lusztig variety for $PGL_3(L)$, we have the same pattern and dimension formulas for $GL_3(L)$.

In a different way, if the automorphism $\sigma : GL_3(L) \rightarrow GL_3(L)$ is replaced by $\sigma' = Ad(\tau) \circ \sigma : GL_3(L) \rightarrow GL_3(L)$, then $GL(L)_3^{\sigma'}$ is the group

of units of the division algebra \mathbb{D}_3 (i.e. \mathbb{D}_3^\times). As it is described in [GHN], the affine Deligne-Lusztig variety $X_{\tilde{w}\tau}(\tau)$ for PGL_3 is isomorphic to the affine Deligne-Lusztig variety $X_{\tilde{w}}(1)$ for the group \mathbb{D}_3^\times . Thus we have the same pattern and dimension formulas for \mathbb{D}_3^\times .

3.4. Rational points for some affine Deligne-Lusztig varieties. In general, if $X_{\tilde{w}}(b) \neq \emptyset$, then it has infinitely many irreducible components. However, if $b \in G(L)$ is superbasic, then $X_{\tilde{w}}(b)$ contains only finite irreducible components and the number of rational points is finite.

Proposition 3.9 (He). *Let $G = PGL_n$ be split over F . If x is a superbasic element in \tilde{W} , then for any $\tilde{w} \in \tilde{W}$*

$$\#X_{\tilde{w}}(x)(\mathbb{F}_q) = nq^{\frac{\ell(\tilde{w})}{2}} f_{\tilde{w}, \mathbb{O}}|_{v=\sqrt{q}},$$

where $x \in \mathbb{O}$ and it is the conjugacy class of \tilde{w} .

Applying Proposition 3.9 to the group $G = PGL_3$, we can obtain an explicit formula.

Corollary 3.10. *Let $G = PGL_3$ be split over F . Let $\tau \in G(L)$, then for any $\tilde{w} \in \tilde{W}$ we have*

$$\#X_{\tilde{w}}(\tau)(\mathbb{F}_q) = \begin{cases} 3q^{\frac{\ell(\tilde{w})}{2}}, & \tilde{w} \in \mathbb{O}_{id, \tau} \\ 3 \lceil \frac{\ell(\tilde{w})-6i+3}{4} \rceil q^{\frac{\ell(\tilde{w})-1}{2}} (q-1), & \tilde{w} \in \mathbb{O}_{i, \tau}, i \in \mathbb{N}_+ \text{ and } \ell(\tilde{w}) \geq 6i-1 \\ 3 \lceil \frac{\ell(\tilde{w})-6i+1}{4} \rceil q^{\frac{\ell(\tilde{w})-1}{2}} (q-1), & \tilde{w} \in \mathbb{O}_{1-i, \tau} \text{ and } \ell(\tilde{w}) \geq 6i+1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It follows directly from Proposition 3.9 and Propositions 2.38, 2.39. \square

3.5. The GHKR conjecture. Before we recall a conjecture of Görtz-Haines-Kottwitz-Reuman, let give some notations first. Let $G(L)$ be as in Chapter 2. For any $b \in G(L)$, we denote by J_b the σ -centralizer of b (i.e. $J_b = \{g \in G(L) \mid gb\sigma(g)^{-1} = b\}$). By definition, the defect of b ($def(b)$) is the F -rank of G minus the F -rank of J_b (i.e. $def(b) = rk_F G - rk_F J_b$). We follow [Ha] Lemma 4.6 to calculate the defect of b .

The Görtz-Haines-Kottwitz-Reuman conjecture is stated as:

Conjecture 3.11. *Let $b \in G(L)$ and b' be a basic element in $G(L)$ such that $\kappa(b) = \kappa(b')$. Then for $\tilde{w} \in \tilde{W}$ with sufficiently large length, $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $X_{\tilde{w}}(b') \neq \emptyset$. In this case,*

$$\dim X_{\tilde{w}}(b) = \dim X_{\tilde{w}}(b') - \langle \nu_b, \rho \rangle + \frac{1}{2}(def(b') - def(b)).$$

Theorem 3.12. *The Conjecture 3.11 is true for the following groups:*

$$GL_3(L), \quad PGL_3(L), \quad U_3(L), \quad \mathbb{D}_3^\times.$$

Proof. By §3.3, it is enough for us to check it for the group $PGL_3(L)$ and $U_3(L)$. By §3.1, we know the emptiness/nonemptiness pattern for basic b' . For nonbasic b , using the Dimension=Degree theorem and class polynomials in Chapter 3, if $\tilde{w} \in \tilde{W}$ with sufficiently large length, we check case-by-case that $X_{\tilde{w}}(b) \neq \emptyset$ if and only if $X_{\tilde{w}}(b') \neq \emptyset$. If they are nonempty, it remains to check the comparison of their dimensions and we check case-by-case as well. For example, let $G(L) = PGL_3(L)$ and $b' = 1$. If b corresponds to \mathbb{O}_{λ_0} , where $\lambda_0 \in P_+ \cap Q_{sh}$. For those $\tilde{w} \in \tilde{W}$ with sufficiently large length, and $X_{\tilde{w}}(b) \neq \emptyset$, we have

$$\dim X_{\tilde{w}}(b) - \dim X_{\tilde{w}}(1) = \frac{1}{2}\ell(\mathbb{O}_{\lambda_0}) - \langle \lambda_0, 2\rho \rangle.$$

Since $\lambda_0 \in P_+$, $\ell(\mathbb{O}_{\lambda_0}) = \ell(t^{\lambda_0}) = \langle \lambda_0, 2\rho \rangle$. By direct calculation, we have $def(1) = def(b)$. Thus

$$\dim X_{\tilde{w}}(b) = \dim X_{\tilde{w}}(1) - \langle \nu_b, \rho \rangle + \frac{1}{2}(def(1) - def(b)).$$

If $b' = \tau$ and b corresponds to $\mathbb{O}_{\lambda_0, \tau}$, where $\lambda_0 \in P_+ \cap Q_{sh}$ and $\lambda_0 \neq (2i - 1)\alpha_1 + i\alpha_2$ for all $i \in \mathbb{N}_+$. Thus we have

$$\dim X_{\tilde{w}}(b) - \dim X_{\tilde{w}}(\tau) = \frac{1}{2}(\ell(\mathbb{O}_{\lambda_0, \tau}) + 2) - \langle \nu_{\mathbb{O}_{\lambda_0, \tau}}, 2\rho \rangle.$$

In this case, we have $\nu_b = \nu_{\mathbb{O}_{\lambda_0, \tau}} = \lambda_0 - \frac{1}{3}(\alpha_1 + 2\alpha_2)$, $\ell(\lambda_0) = \ell(\mathbb{O}_{\lambda_0, \tau}) + 2$, and $def(\tau) - def(b) = 2$, thus

$$\dim X_{\tilde{w}}(b) = \dim X_{\tilde{w}}(\tau) - \langle \nu_b, \rho \rangle + \frac{1}{2}(def(\tau) - def(b)).$$

For other cases, the methods are quite similar and we will omit them. \square

3.6. Affine Deligne-Lusztig varieties in the affine Grassmannian. We recall the affine Deligne-Lusztig variety in the affine Grassmannian first. We keep the notations as in Section §1. Let \mathbb{G} be the smooth affine group scheme associated with the special vertex of the Bruhat-Tits building of G . We denote by $L^+\mathbb{G}(R) = \mathbb{G}(R[[\epsilon]])$ the infinite dimensional affine group scheme. The twisted *affine Grassmannian* is defined by the fpqc quotient $\mathbf{Gr} = LG/L^+\mathbb{G}$. We have the Cartan decomposition

$$G(L) = \bigsqcup_{\mu \in P_+} L^+\mathbb{G}(\mathbf{k})\epsilon^\mu L^+\mathbb{G}(\mathbf{k}), \quad Gr(\mathbf{k}) = \bigsqcup_{\mu \in P_+} L^+\mathbb{G}(\mathbf{k})\epsilon^\mu L^+\mathbb{G}(\mathbf{k})/L^+\mathbb{G}(\mathbf{k}).$$

Definition 3.13. Let $b \in G(L)$ and $\mu \in P_+$, the affine Deligne-Lusztig variety $X_\mu(b)$ in the affine Grassmannian Gr is defined by

$$X_\mu(b)(\mathbf{k}) = \{g \in G(L) \mid gb\sigma(g)^{-1} \in L^+\mathbb{G}(\mathbf{k})\epsilon^\mu L^+\mathbb{G}(\mathbf{k})/L^+\mathbb{G}(\mathbf{k}).\}$$

Let w_0 be the longest element in W . We notice that any $W \times W$ -coset of \tilde{W} contains a unique maximal element and this element is of the form $w_0 t^\lambda$ for some $\lambda \in P_+$. An element in this double coset is of the form $xt^\lambda y$ for $x \in W$ and $y \in {}^{I(\lambda)}W$. Here ${}^{I(\lambda)}W = \{s_i \in \mathbb{S} \mid \langle \lambda, \alpha_i \rangle = 0\}$. Let $\tilde{w} = w_0 t^\lambda$, and $\lambda \in P_+$. For the special element $w_0 t^\lambda$, we have

Theorem 3.14 (He). *Let $\lambda \in P_+$, $x \in W$ and $y \in {}^{I(\lambda)}W$. For any $b \in G(L)$,*

$$\dim X_{xt^\lambda y}(b) \leq \dim X_{w_0 t^\lambda}(b) - \ell(w_0) + \ell(x).$$

What's more, for the special element $w_0 t^\lambda$, there is a complete answer for the emptiness/nonemptiness pattern and dimension formula (see [He3] Theorem 6.1).

3.7. Leading coefficient of $f_{w_0 t^\lambda, \mathbb{O}}$. Let $p : \mathbf{Fl} \rightarrow \mathbf{Gr}$ be the projection. Each fiber of p is isomorphic to $L^+ \mathbb{G}/I$, which is of dimension $\ell(w_0)$. Since $p^{-1}X_\lambda(b) = \bigsqcup_{\tilde{w} \in W t^\lambda W} X_{\tilde{w}}(b)$ and Theorem 3.14, for any $\tilde{w} \in W t^\lambda W$ we have $\dim X_{\tilde{w}}(b) \leq \dim X_{w_0 t^\lambda}(b)$. Based on information of class polynomials and the reduction method, it is expected that the irreducible components of $X_{\tilde{w}}(b)$ of maximal dimension are controlled by the leading coefficient of the corresponding class polynomial $f_{w_0 t^\lambda, \mathbb{O}}$. And we denote by $f_{\tilde{w}, \mathbb{O}_b}$ the class polynomial corresponding to $X_{\tilde{w}}(b)$, and it is indicated by the ‘‘Dimension=Degree’’ theorem.

Instead of considering the irreducible components of $X_{\tilde{w}}(b)$ of maximal dimension, we are going to consider the leading coefficient of the corresponding class polynomial.

Proposition 3.15. *Given $\tilde{w} \in \tilde{W}$, we denote N_0 to be the leading coefficient of $f_{\tilde{w}, \mathbb{O}_2}$. We denote $L(f_{\tilde{w}, \mathbb{O}})$ to be the leading coefficient of $f_{\tilde{w}, \mathbb{O}}$. Moreover, we assume that $\tilde{w} = w_0 t^\lambda$, where $\lambda \in P_+$.*

(1) *If $\lambda = k_0(\alpha_1 + \alpha_2)$ with $k_0 \in \mathbb{N}$ and it is large enough, then*

$$L(f_{\tilde{w}, \mathbb{O}_b}) = \begin{cases} N_0 - 2i, & b \longleftrightarrow \mathbb{O}_{i\alpha_1 + 2i\alpha_2} \text{ or } \mathbb{O}_{2i\alpha_1 + i\alpha_2}; \\ N_0 - i, & b \longleftrightarrow \mathbb{O}_\lambda, \lambda \in \{i(\alpha_1 + \alpha_2)\} \sqcup E'_{i(\alpha_1 + \alpha_2)}, \text{ or} \\ & b \longleftrightarrow \mathbb{C}_i \text{ or } \mathbb{C}'_i. \end{cases}$$

Here $i \in \mathbb{N}$ and k_0 relates to i is quite large.

(2) *If $\lambda = i_0(\alpha_1 + 2\alpha_2) + k_0(\alpha_1 + \alpha_2)$ with $k_0 \in \mathbb{N}$ and it is large enough, then*

$$L(f_{\tilde{w}, \mathbb{O}_b}) = \begin{cases} N_0, & b \longleftrightarrow \mathbb{O}_{i_0\alpha_1 + 2i_0\alpha_2}, \text{ or } \mathbb{C}_i, \mathbb{C}'_i (i \leq i_0); \\ N_0 - j, & b \longleftrightarrow \mathbb{O}_\lambda, \lambda \in \{(i_0 + j)\alpha_1 + (2i_0 + j)\alpha_2\} \sqcup E_{(i_0 + j)\alpha_1 + (2i_0 + j)\alpha_2} \\ & \sqcup E'_{(i_0 + j)\alpha_1 + (2i_0 + j)\alpha_2}, \text{ or} \\ & b \longleftrightarrow \mathbb{C}_{i_0 + j} \text{ or } \mathbb{C}'_{i_0 + j}. \end{cases}$$

(3) *If $\lambda = i_0(2\alpha_1 + \alpha_2) + k_0(\alpha_1 + \alpha_2)$ with $k_0 \in \mathbb{N}$ and it is large enough, then*

$$L(f_{\tilde{w}, \mathbb{O}_b}) = \begin{cases} N_0, & b \longleftrightarrow \mathbb{O}_{2i_0\alpha_1 + i_0\alpha_2}, \text{ or } \mathbb{C}_i, \mathbb{C}'_i (i \leq i_0); \\ N_0 - j, & b \longleftrightarrow \mathbb{O}_\lambda, \lambda \in \{(2i_0 + j)\alpha_1 + (i_0 + j)\alpha_2\} \sqcup E_{(2i_0 + j)\alpha_1 + (i_0 + j)\alpha_2} \\ & \sqcup E'_{(2i_0 + j)\alpha_1 + (i_0 + j)\alpha_2}, \text{ or} \\ & b \longleftrightarrow \mathbb{C}_{i_0 + j} \text{ or } \mathbb{C}'_{i_0 + j}. \end{cases}$$

Proof. We check (1) and the others are similar. If b corresponds to $\mathbb{O}_{i\alpha_1 + 2i\alpha_2}$ (or $\mathbb{O}_{2i\alpha_1 + i\alpha_2}$), where $i \in \mathbb{N}_+$. Then $\mathbb{O}_b = \mathbb{C}_{2i}$ (or $\mathbb{O}_b = \mathbb{C}'_{2i}$). In this case, $f_{\tilde{w}, \mathbb{O}_2} = N_0(v - v^{-1})$ and $f_{\tilde{w}, \mathbb{C}_{2i}} = f_{\tilde{w}, \mathbb{C}'_{2i}} = (N_0 - 2i)(v - v^{-1})^2$. If b corresponds to \mathbb{O}_{λ_0} where $\lambda_0 = i(\alpha_1 + \alpha_2)$ (or $\lambda_0 \in E_{i(\alpha_1 + \alpha_2)} \sqcup E'_{i(\alpha_1 + \alpha_2)}$), then $\mathbb{O}_b = \mathbb{O}_{\lambda_0}$, and $f_{\tilde{w}, \mathbb{O}_{\lambda_0}} = (N_0 - i)(v - v^{-1})^3 + (v - v^{-1})$ (or $f_{\tilde{w}, \mathbb{O}_{\lambda_0}} = (N_0 - i)(v - v^{-1})^3$).

If b corresponds to \mathbb{C}_i or \mathbb{C}'_i , then $\mathbb{O}_b = \mathbb{C}_i$ or $\mathbb{O}_b = \mathbb{C}'_i$, and $f_{\tilde{w}, \mathbb{C}_i} = f_{\tilde{w}, \mathbb{C}'_i} = (N_0 - i)(v - v^{-1})^2$. Thus (1) is proved. \square

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CURRENT: BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, No.5 YIHEYUAN ROAD, HAIDIAN DISTRICT, BEIJING 100871, CHINA
E-mail address: yzw@connect.ust.hk